SEARCHING FOR SELF-DUALITY IN NON-MAXIMALLY SUPERSYMMETRIC BACKGROUNDS

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Dedicated to my husband, for never tiring; always inspiring me and for sharing an unending enthusiasm for the progression of scientific knowledge and the philosophy of science. Thank you for your support. Thank you for teaching me.

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Abstract

Fermionic T-duality is the generalisation to superspace of bosonic T-duality (i.e. to include fermionic degrees of freedom). Originally, T-duality described the equivalence relation between two physical theories, each living on a different background. However, this thesis is concerned with fermionic T-duality and its role in self-duality. The goal is to determine whether AdS backgrounds with less than maximal supersymmetry are self-dual. A background is said to be self-dual if, after a specific sequence of bosonic and fermionic T-duality transformations, the original background is recovered. Self-dual backgrounds are of great interest due to their link to integrability. Fermionic T-duality has played a pivotal role in proving that the maximally supersymmetric background $AdS_5 \times S^5$ is self-dual. This background is also known to be integrable, therefore, when it was shown to be self-dual, the hypothesis that self-duality implied integrability, and vice-versa, was born. We investigate how far this hypothesis may be stretched for a number of AdS backgrounds, for which integrability has already been determined. The following backgrounds were considered: $AdS_2 \times S^2 \times T^6$ and $AdS_d \times S^d \times T^{(10-3d)}$ $(d = 2, 3)$. This question of self-duality was approached in two ways. In the first approach we show that these less supersymmetric backgrounds are self-dual by working with the supergravity fields and using the fermionic Buscher procedure derived by Berkovits and Maldacena. In the second approach, we verify the self-duality of Green-Schwarz supercoset σ-models on $AdS_d \times S^d$ $(d = 2, 3)$ backgrounds. Furthermore, we prove the self-duality of $AdS_5 \times S^5$ without gauge fixing κ-symmetry. We show that self-duality is a property which holds for the exceptional backgrounds, where the need to T-dualise along one of the spheres arises, again. Nature is not supersymmetric, therefore learning how to do physics in $AdS_5 \times S^5$ is not enough. In order to understand theories like Quantum Chromodynamics, we need to systematically break the supersymmetry present in our toy models. In this regard, it is easy to appreciate the significance of studying backgrounds with less than maximal supersymmetry.

Declaration

The work presented in this thesis is partly based on research in collaboration with Associate Professor Jeff Murugan and Dr Michael Abbott at The Laboratory for Quantum Gravity & Strings, University of Cape Town. Our other important collaborators include: Associate Professor Silvia Penati (University of Milan-Bicocca), Antonio Pittelli (University of Surrey), Professor Dmitri Sorokin (University of Padova), Dr Per Sundin (University of Milan-Bicocca), Professor Martin Wolf (University of Surrey) and Dr Linus Wulff (Imperial College). The list below identifies sections which are partly based on the listed publications:

Chapter 3: M. C. Abbott, J. Tarrant & J. Murugan, *Fermionic T-Duality of* $AdS_n \times S^n (\times S^n) \times T^m$ *using IIA Supergravity*, Class. Quant. Grav. **33** (2016), 075008, arXiv:1509.07872 [hep-th]

Chapter 4: M. C. Abbott, J. Murugan, S. Penati, A. Pittelli, D. Sorokin, P. Sundin, J. Tarrant, M. Wolf & L. Wulff, *T-Duality of Green-Schwarz Superstrings on* $AdS_d \times S^d \times M^{(10-2d)}$, JHEP **1512** (2015), 104, arXiv:1509.07678 [hep-th]

I hereby declare that this thesis has not been submitted, either in the same or different form, to this or any other university for a degree and that it represents my own work.

Signed by candidate .... 21 March 2017

Author

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Part I

Introduction and Overview

Introduction

String theory is the leading candidate for a theory which unifies gravity and quantum mechanics, the research efforts in this exciting field of physics have been remarkable. String theory has undergone enormous developments to arrive at the current paradigm - where symmetries play an even larger role in our understanding of string theory [1]. As a result of this intensive development, the field has exploded in a number of directions. Consequently, it has become virtually impossible to keep up to date with all the streams of research being explored. This is both very exciting and hugely daunting, especially for newcomers. That so many young scientists still enter this field, despite the steep learning curve, speaks volumes about the enticing problems that still need to be considered and solved.

During the mid-nineties string dualities suddenly formed the basis of serious study. Finally, there was a general understanding of the importance of these types of symmetries. Duality symmetries do not manifest themselves perturbatively in the weak coupling expansion of string theory, the context in which string theory has mostly been probed, instead they give us information about the exact string theory which is more complicated to study [1]. As a major driving force in string theory, the uncovering of non-perturbative duality symmetries, asks whether it is possible to study non-perturbative physics using perturbative techniques - a core tenet of the new paradigm. Hence, it is not too surprising that string dualities remained hidden for so long. After gaining access to strong coupling aspects through the use of mathematical tools associated with string duality, it became clear that the relationship between weak and strong coupling would be important. What could strong coupling teach us about weak coupling? Interest turned towards the idea of weak/strong duality. In string theory this means that as the string coupling g_s becomes large we are able to find new dual degrees of freedom whose fluctuations become small. That is, they are described by a new coupling $g_s' \sim 1/g_s$. String theory includes gauge theory, hence we need to consider a further point: weak/strong duality in the gauge theory is a necessary (but not sufficient) condition to ensure that weak/strong duality exists in the string theory [1]. Progress was made when Hull, Townsend and independently Witten [2–4] proposed an almost complete set of duals to all string theories possessing at least $\mathcal{N} = 4$ supersymmetry. Suddenly, the idea of duality had been elevated to a general principle which could be applied to all string theories in any number of dimensions where previously, the idea of duality was only an isolated set of conjectures. These results led to the increasing prominence of weak/strong coupling duality.

String dualities now play a central role in modern string theory research [2,4–6]. Since the term duality is used in many contexts throughout various fields in physics, it is necessary to clarify what we mean by it here. A duality is a map, which is often invertible[1], between two theories which sends states in one theory to states in the other theory. At the same time, interactions, amplitudes, and global symmetries are usually preserved [7]. The two theories that are found to be dual may be thought of as being physically equivalent. Dualities are particularly important as they are used to map difficult problems into simpler ones, which we may have more chance of solving. Initially, five consistent superstring theories in 10-dimensions were known. These were the type I, two type II, and two heterotic string theories. Additionally, there is also an 11-dimensional theory known as M-theory. The different string theories are distinguished from each other by the number (and kind) of supersymmetries and also by the various worldsheet topologies permitted (i.e. oriented versus unoriented). Although the five string theories appear to be very different when dealt with in the weak-coupling regime, we now know that they are all related to each other by one of the known string dualities. The connection between the different theories is known as the 'web of string dualities', as seen in Figure 1.

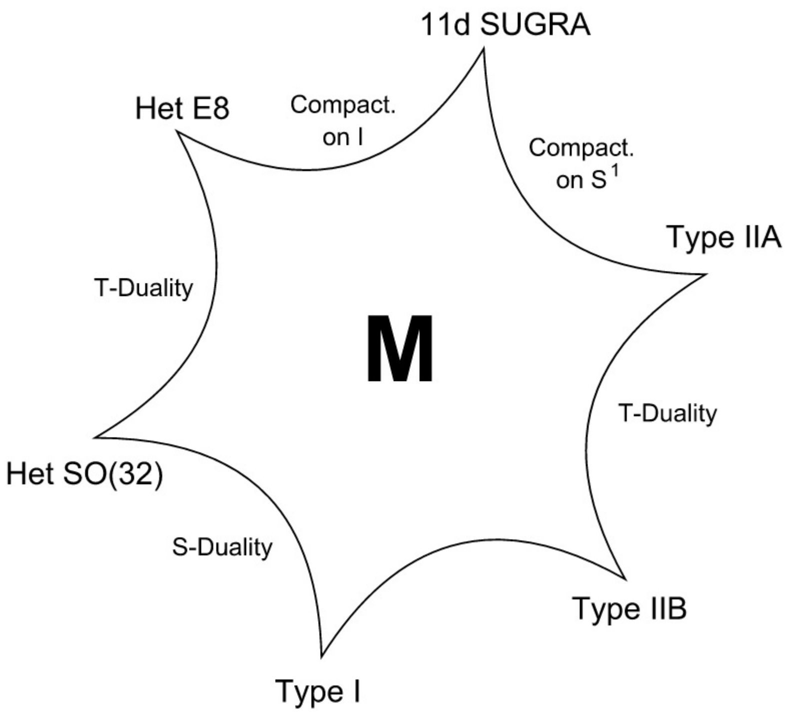

Figure 1: The web of string dualities [8] where Type I string theory is related to type II string theory via a bosonic T-duality.

In this thesis, our focus is on one particular string duality called T-duality and its many facets. In general, it relates two theories with different spacetime geometries. The two theories are considered to be equivalent in the sense that all observable quantities in one description are identified with observable quantities in the dual description. T-duality differs from other dualities in that it does not mention the exchange between weak and strong coupling of the two

[1] As is the case for Abelian T-duality, however, this is not true for non-Abelian T-duality

dual theories. However, the AdS/CFT correspondence which was discovered two decades ago, is a duality which precisely deals with the weak/strong coupling concept described earlier. The AdS/CFT correspondence (or gauge/gravity du- ality) relates a quantum field theory and gravity. For example, this correspondence is well understood in the case of the equivalence between $\mathcal{N} = 4$ super Yang-Mills and type IIB strings on $AdS_5 \times S^5$ [6]. Put another way, the correspondence relates the quantum physics of strongly correlated many-body systems to the classical dynamics of gravity in one dimension higher [9].

In this context, type IIA and type IIB string theories are related by bosonic T-duality, the two being exchanged with each other under this duality transformation. The same is true for the two heterotic string theories, $E_8 \times E_8$ and $SO(32)$. A central idea in string theory, concerning dualities, is that the strongly coupled limit of *any* string theory is equivalent to the weakly coupled limit of some other string theory [1]. All the connections between various string theories embody this principle. It must be noted that Figure 1 is an oversimplification because the different limits are characterized by more than just different string theories, but also by the different topologies of the various compact dimensions [1].

T-duality is one of the oldest and simplest examples of a string duality [10], being a manifestation of the extended nature of strings. T-duality takes a non-linear σ-model (which may be described by the Polyakov action) with a target space possessing an isometry and, through the application of the Buscher procedure [11–13], returns a T-dual σ-model, after gauging the isometry and integrating out the gauge fields. This worldsheet approach[2] to deriving T-duality in string theory, via the Buscher procedure, is now standard practice. T-duality was discovered as a symmetry of the effective potential in the compactification associated with radial inversion symmetry: $R \leftrightarrow \alpha'/R$, where R is the radius of compactification on the spacetime [17, 18]. The Buscher procedure was initially applied to Abelian isometries, but further studies made it possible to extend the application to non-Abelian isometries and fermionic isometries [19–22, 84].

Indeed it is the fermionic generalisation that we are concerned with in this thesis. Motivation for fermionic T-duality arose when hidden symmetries, not manifest in the Lagrangian, were discovered in the scattering amplitudes of supersymmetric theories [23]. Soon after this, a remarkable connection between planar [3] scattering amplitudes and Wilson loops in $\mathcal{N} = 4$ super Yang-Mills was found [26–28], see [29] for a review. The most widely studied example of the AdS/CFT correspondence [6] is 4-dimensional $\mathcal{N} = 4$ super Yang-Mills and type IIB superstring theory on $AdS_5 \times S^5$. The planar limit of super Yang-Mills theory corresponds to the classical limit in string theory. The scattering amplitudes are described in terms of scalar one-loop integrals where, in the planar limit, contributing integrals reveal an interesting property. That is, when they are exchanged for their dual graphs they appear to exhibit a

[2]Originally worldsheet techniques [14, 15] had dominated the scene. Currently the trend is toward the spacetime approach taking over [16].

[3]The planar limit [24], for which the t'Hooft coupling $\lambda = g_{YM}^2 N$ is held fixed, is the case where only planar diagrams survive [25]. This occurs because the genus expansion in terms of Feynman diagrams corresponds to the $1/N$ expansion.

novel, non-trivial conformal symmetry, called dual conformal symmetry [30,31]. Originally, this symmetry showed up in the perturbative computations explored in [32], though soon afterwards dual conformal symmetry was found in next to next to maximally-felicity-violating (next to MHV) amplitudes [31], where it was then extended to full dual superconformal symmetry.

Ordinary conformal symmetry in the T-dual model is the dual conformal symmetry in the original model. To make sense of their relationship we note that dual conformal symmetry of scattering amplitudes in the gauge theory acts on momenta in much the same way as ordinary conformal symmetry acts on coordinates in the dual model. Simultaneously, it associates to each scattering amplitude a string worldsheet in a dual AdS space [33–35]. Berkovits and Maldacena showed that dual superconformal symmetry may be understood using a T-duality symmetry of the full superstring theory on $AdS_5 \times S^5$. The T-duality involves ordinary bosonic T-dualities (considered in [26]) and a novel set of fermionic T-dualities. Application of these symmetries returned a background equivalent to $AdS_5 \times S^5$. This resultant background was termed *self-dual*.

Fermionic T-duality generates new solutions via the identification of commuting fermionic isometry directions built from Killing spinors [22]. We may attribute to the existence of dual superconformal symmetry, the self-duality of the superstring σ-model under a sequence of T-duality transformations [36]. These transformations are of the bosonic and fermionic string modes on the worldsheet corresponding to some commuting isometries of $AdS_5 \times S^5$, see [21,22] for details. Simultaneously, self-duality arises as a result of the combined bosonic and fermionic T-dualities not changing the form or values of the background fields on $AdS_5 \times S^5$, such as the dilaton and the Ramond-Ramond flux [23]. The dilaton and metric tensor make up the NSNS (Neveau-Schwarz-Neveau-Schwarz) sector in the RNS (Ramond-Neveau-Schwarz) formalism of type II supergravity. The dilaton is a scalar field related to the string coupling g_s . The RR sector is made up of the Ramond-Ramond fields, which are differential forms, that act as the higher dimen- sional generalizations of the fields in Maxwells electromagnetism. Fermionic T-duality, dual superconformal symmetry and their relationship to each other have been broadly studied in the context of the maximally supersymmetric $AdS_5 \times S^5$ background [21–23]. Therefore, this example is well understood. On the other hand, the fermionic T-duality of σ-models for superstrings on integrable[4] AdS backgrounds possessing less than maximal supersymmetry is much less understood. Therefore, we are concerned with extending the understanding of this idea to examples possessing less supersymmetry.

The readers first question may well be: Why would one study the less than maximally supersymmetric backgrounds? Furthermore, what could we learn? What we know is that the AdS/CFT correspondence for AdS_5/CFT_4 with $AdS_5 \times S^5$ has been studied in great detail. Much has been learned as a result and shortly after AdS/CFT was discovered ideas started drifting toward unfamiliar territory with regards to this duality [42]. This created a platform for testing the generality of this idea. To test generality, a good question is: If we know the results for

[4]As mentioned in the footnote found in [36], the classical integrability of superstrings in AdS backgrounds has been studied in [37–41].

maximal supersymmetry, then what can we infer for less supersymmetry? Do we have reason to believe that what we have observed so far, will hold for less than maximally supersymmetric cases? Following this logic, it makes sense to choose a background with less supersymmetry, but perhaps with as little supersymmetry broken as possible. Such a next-to-maximally supersymmetric background would be $AdS_4 \times \mathbb{C}P^3$ associated with N = 6 Cherns-Simons (ABJM).

As for the case of $\mathcal{N} = 4$ super Yang-mills, integrability features in ABJM in the planar limit [42–44] and in string theory on $AdS_4 \times \mathbb{C}P^3$ [38, 45–47]. There is also an overwhelming amount of evidence provided by perturbative calculations, that amplitude/Wilson loop duality and dual superconformal symmetry are present for the AdS_4/CFT_3 partnership. It is remarkable that such similarity between AdS_5/CFT_4 and AdS_4/CFT_3 cases exist and it is for this reason that string theorists were led to hope that a self-dual mapping of the geometry $AdS_4 \times \mathbb{C}P^3$ under bosonic and fermionic T-duality would account for the observed perturbative symmetries of the ABJM theory. We have hope because $AdS_5 \times S^5$ is self-dual under these exact T-duality transformations. Unfortunately, all attempts to prove self-duality under bosonic and fermionic T-duality have failed. Failed, despite the important ingredients like integrability remaining intact. We would like to work out part of the puzzle that is: Why this failure? To this end we test the self-duality of backgrounds which are integrable (like $AdS_4 \times \mathbb{C}P^3$ and $AdS_5 \times S^5$) but possess less supersymmetry (like $AdS_4 \times \mathbb{C}P^3$).

Whether or not self-duality is present in the backgrounds considered herein, we will be one step closer to understanding the issues surrounding $AdS_4 \times \mathbb{C}P^3$. Furthermore, we know that for $AdS_5 \times S^5$, T-self-duality appears to be a predictor of integrability. This is a brilliantly useful relationship as self-duality is much simpler to demonstrate than integrability. Therein, we find more motivation to study $AdS_4 \times \mathbb{C}P^3$. Will this predictor of integrability be completely ruled out? An interesting question indeed! Therefore, we take it upon ourselves to see how far the relation (self-duality $\implies$ integrability) extends by considering backgrounds with less than maximal supersymmetry. In summary: Does less supersymmetry account for the failure of $AdS_4 \times \mathbb{C}P^3$ being self-dual?

Overview of Thesis

This thesis is divided into three Parts and further into Chapters. **Part I: Introduction and Overview** contains the general introduction to the whole thesis, as well as Chapter 1 and Chapter 2 which contain the essential mathematical framework required to understand the work which follows. **Part II: Fermionic T-duality of Non-maximally Supersymmetric Backgrounds** contains the body of original work created as part of two collaborations. The first team is based at the The Laboratory for Quantum Gravity & Strings at the University of Cape Town.

Chapter 1 provides an introduction to bosonic T-duality. We discuss some historical features to provide context and then go on to derive the Buscher rules. Two cases are examined: the original Abelian T-duality and its generalisation to non-Abelian isometries. We also consider generalised geometry and the $O(D,D)$ group which encompasses Abelian T-duality transformations.

Chapter 2 introduces fermionic T-duality, the fermionic generalisation of ordinary T-duality. It is this idea which forms the basis of this thesis. We discuss its properties, its differences compared to ordinary T-duality and then go on to derive the Buscher procedure for fermionic isometries. A general prescription for carrying out this procedure is also given.

Chapter 3 considers the self-duality of the backgrounds $AdS_2 \times S^2 \times T^6$ and $AdS_d \times S^d_+ \times S^d_- \times T^{10-3d}$ for $d = 2, 3$. We use the Buscher procedure and work at the level of the supergravity fields to show self-duality. This approach is very useful for seeing how the Ramond-Ramond fields transform explicitly. After studying particular cases we end with a general argument which pulls the ideas in this chapter together. We start in type IIA string theory, although we can easily arrive at a type IIB theory through a bosonic T-duality.

Chapter 4 deals with supercoset σ-models. We verify the self-duality of such models on $AdS_d \times S^d$ backgrounds for $d = 5$ under combined bosonic and fermionic T-dualities without fixing κ-symmetry gauge.

Chapter 5 We prove that the exceptional backgrounds $AdS_d \times S^d_+ \times S^d_-$ for $d = 2, 3$ are self-dual. This approach is to be contrasted with the methods in Chapter 3.

Chapter 1

Introductory Bosonic T-duality

1.1 Introduction

The string theories that we are interested in studying are described by non-linear σ-models. These tell us how to embed a 2-dimensional manifold into a target spacetime, the goal being to embed the manifold into a spacetime of physical interest. This 2-dimensional manifold is called the worldsheet which is the natural generalization of the worldline for a particle. The worldsheet is the minimal surface interpolating between the given initial and final configurations of the string. The 1-dimensional extended nature of the string has some interesting consequences. In particular, we are interested in the unique duality symmetry called target space duality, or just T-duality [1]. Duality symmetries provide a one-to-one correspondence (or map) between two different systems. This map furnishes a dictionary which allows us to translate between the two theories in a precise way. Often, through this dictionary we are able to map difficult problems into easy ones, and vice versa [2]. This chapter deals with bosonic T-duality which is the original form of this duality symmetry. Interest in bosonic T-duality has been revived due to the discovery of its cousin, fermionic T-duality [48], to be explored in the following chapter. The bosonic Neveu-Schwarz (NS) sector is the massless sector common to all types of string theory. It contains a symmetric 2-form tensor field G called the metric, an antisymmetric 2-form tensor field B called the Kalb-Ramond field and the scalar field Φ called the dilaton. The B field is a higher rank tensor analogue of a gauge field. The dilaton is related to the string coupling g_s as follows

$$g_s = \exp(\langle\Phi\rangle)$$

[1]The 'T' also stands for toroidal. A compactification is toroidal if the compact space is a torus.

[2]Note that symmetries always operate within the same system, leading to conservation laws which simplify problems greatly, but they are quite different from dualities in that they do not map difficult problems into easy ones. Instead, symmetries simplify difficult problems by applying constraints to the possible forms of the answer.

where $\langle\Phi\rangle$ is the vacuum expectation value of Φ. The coupling is a dynamical variable in string theory. T-duality, for closed strings, relates small distance scales to large distance scales. It is also crucial to the understanding of D-branes, which arise in open string theory [49]. An important point to mention is that T-duality maps $R \to 1/R$ and consequently it is the winding of strings that is crucial [3]. Point particles are 0-dimensional and thus cannot possess winding modes. Thus it is this extended nature of strings which exposes new properties such as T-duality. This chapter introduces bosonic T-duality in the Abelian and non-Abelian contexts. We place emphasis on closed string theory throughout this thesis.

1.2 Abelian T-Duality

Begin by considering a closed bosonic string [4] which moves in the background $S^1 \times R^{1,24}$. This means that the spacetime is compactified along one direction, i.e. the S^1. This circle has a radius R. The coordinates X describe the embedding of the worldsheet into spacetime, that is, our coordinates are parametrised by the worldsheet coordinates (τ, σ), much like the worldline is parametrized by proper time. Specifically, the circle compactification amounts to singling out one direction, say $X^1(\tau, \sigma)$, which is compact. Here τ is the time on the worldsheet and σ is the circular, periodic direction along the closed string. This compactification changes the string dynamics in two important ways [51]. Firstly, the spatial momentum, p^1 , may no longer assume any value. It is now quantized:

$$p^1 = \frac{n}{R}, \quad n \in \mathbb{Z}.$$

Secondly, the boundary conditions for the mode expansion of X become more general after compactification. As a result, we are able to move around the string whilst relaxing the condition $X(\tau, \sigma + 2\pi) = X(\tau, \sigma)$ to $X^1(\tau, \sigma + 2\pi) = X^1(\tau, \sigma) + 2\pi mR$, where $m \in \mathbb{Z}$ and we have an isometry along the X^1-th direction. Here the integer m tells you how many times the string winds around the compact direction, S^1. Naturally, m is called the winding number, that is, the number of times X^1 (the compact direction) needs to be circled in order to return to the starting point. This winding term is a uniquely stringy phenomenon. To find the energy spectrum, consider the periodic field X^1, which has the following mode expansion

$$X^1(\tau, \sigma) = x^1 + \frac{\alpha' n}{R}\tau + mR\sigma + \text{oscillator modes},$$

incorporating both quantized momentum and the possibility of winding number. The centre of mass for the system is given by $x^1 + \alpha Rn\tau$. Start with a simple non-linear σ- model,

[3] Here $1/R$ is the radius of the dual background in terms of the radius R of the original background.

[4] The worldsheet approach, in a general framework, has been studied for closed bosonic strings in [50].

$S(G = \delta, B = 0)$, whose dynamics is governed by a wave equation. Here δ is the Kronecker-delta and we have a constant dilaton. The wave equation is much simpler to solve if the coordinates are split into left- and right-moving modes:

$$X^1(\sigma, \tau) = X^1_L(\sigma^+) + X^1_R(\sigma^-)$$

where

$$X^1_L(\sigma^+) = \frac{1}{2}x^1 + \frac{1}{2}\alpha' p_L \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}^1_n \exp(-in\sigma^+)$$
$$X^1_R(\sigma^-) = \frac{1}{2}x^1 + \frac{1}{2}\alpha' p_R \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha^1_n \exp(-in\sigma^-)$$

and α' is the square of the fundamental string scale, l_s . The Fourier modes $\tilde{\alpha}^1_n$ and α^1_n act as creation and annihilation operators respectively. The momenta are given by

$$p_L = \frac{n}{R} + \frac{mR}{\alpha'}, \quad p_R = \frac{n}{R} - \frac{mR}{\alpha'}$$

with $\sigma^{\pm} = \tau \pm \sigma$. Finally, consider the mass spectrum, calculated using

$$M^2 = \sum_{\mu=0}^{25} p_\mu p^\mu.$$

Then the mass squared of the $(N, \tilde{N})$ excited state is given by [51]

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2), \tag{1.1}$$

where N and $\tilde{N}$ are the number operators for left-moving and right-moving oscillation modes of the string, respectively. The first term in (1.1) tells us that a string with $n > 0$ units of momentum around the compact direction, X^1 , contributes n/R to its mass. The second term says that a string winding $m > 0$ times around the compact direction picks up a contribution of $2\pi mRT$ to its mass, where

$$T = \frac{1}{2\pi\alpha'}.$$

Notice that the spectrum (1.1) is invariant under the following transformation

$$m \leftrightarrow n, \quad R \leftrightarrow \frac{\alpha'}{R}.$$

Thus, the strings compactified on circles with radii R and αR are T-dual, and have identical mass spectra. This is called radial inversion symmetry. The precise mapping is given by

$$X^1_R \leftrightarrow -X^1_R, \quad X^1_L \leftrightarrow X^1_L.$$

This extends the symmetry of the spectrum to a symmetry of the whole theory by reflecting [5] the

[5] We actually mean parity reflection here. In non-flat backgrounds possessing an isometry, T-duality does not reduce to a parity transformation any longer, which acts on right-moving (or left-moving) worldsheet variables alone. Instead, in the general case, T-duality acts as a canonical transformation affecting both the left- and

right-moving co-ordinate and leaving the left-moving part invariant [6]. The above procedure also applies to toroidal compactifications [56], specifically to Narain compact-ifications [57], leading to the T-duality group $O(D, D, Z)$. We can think of this group as follows, consider the group $O(2D)$ which has parity and rotations of $2D$ coordinates. Then the group $O(D, D)$ results from still maintaining parity, but letting D coordinates in the metric acquire negative signs. Then the object preserved is the metric with an equal number of negative coordinates as positive. This describes a double space where all D coordinates are doubled, mimicking a string theory with all dimensions being compact [7].

Transformation of Ramond-Ramond Fields

Up till this point, we have been discussing 26-dimensional bosonic string theory. Now we concentrate on the type II 10-dimensional superstring theories. In addition to the NS sector, superstring theory also contains a *Ramond-Ramond* or RR sector, which contains fields which are differential forms. These fields exist in the $10D$ spacetime of type II supergravity which is the classical limit of type II string theory. The RR *p-forms* generalize Maxwell's electromagnetism. The flux transformations for the RR fields can be found in [52]. We start by constructing a *poly-form* composed of RR forms contracted with Γ-matrices [8]. For a IIA theory we have even fluxes:

$$P = \frac{e^{\Phi}}{2} \sum_{n=0}^{5} \not{F}_{2n}, \tag{1.2}$$

where $\not{F}_p = \frac{1}{p!} \Gamma_{\mu^1 \dots \mu_p} F_p^{\mu^1 \dots \mu_p}$ and the fluxes are contracted with the 10-dimensional Γ-matrices. For a IIB theory we take all the odd fluxes:

$$P = \frac{e^{\Phi}}{2} \sum_{n=0}^{4} \not{F}_{2n+1}. \tag{1.3}$$

After T-dualising, the left and right moving modes couple to two distinct sets of vielbeins. However, as a result of these vielbeins describing precisely the same geometries, we may relate them by a Lorentz transformation, Λ. This Λ is used to define an action on spinors parametrized by the matrix Ω satisfying [58]

$$\Omega^{-1} \Gamma^a \Omega = \Lambda^a_b \Gamma^b.$$

For $SU(2)$ isometries this is solved by

$$\Omega = \frac{\alpha'}{\sqrt{G_{99}}} \Gamma_{11} \Gamma_9,$$

right-movers on the worldsheet [52–55].

[6] This reflection applies to fermionic co-ordinates for the superstring [56] as well. Particularly, reflection of the right-moving fermion in the compact direction causes a change in chirality resulting in type IIA and type IIB superstring theories being interchanged.

[7] See the box titled "T-duality in Field Theory" for more information.

[8] See Appendix A for conventions.

with Γ^{11} being the 10-dimensional chirality operator. Then we obtain the dual fluxes through the inversion of Ω. They are related to the original fluxes as follows

$$\tilde{P} = P \cdot \Omega^{-1}.$$

1.2.1 The Buscher Rules

String theory backgrounds are described by both their metric and Kalb-Ramond field, therefore in what follows we refer to (G, B) as the *background*. The action of T-duality re-lates the backgrounds (G, B) and (g, b), which are completely different, using the Buscher rules [12, 13]. Backgrounds related in this way are said to describe the same physics. That is, $S(G, B)$ and $S(g, b)$ give rise to identical theories, furthermore they give rise to the same quantum theory [59]. This may be illustrated by constructing an intermediate action obtained by gauging the isometries [9]. That is, we make the global symmetry a local symmetry through the introduction of a gauge field and the replacement of ordinary derivatives by covariant derivatives. The σ-model is described by the Polyakov action for a bosonic string [60] in conformal gauge. Moving to conformal gauge involves selecting coordinates for which the metric becomes diagonal. It is given by

$$S = \int d^2z[G_{mn}(X) + B_{mn}(X)]\partial X^m \bar{\partial} X^n$$

and written in terms of the complex worldsheet co-ordinate $z = \frac{1}{\sqrt{2}}(\tau + i\sigma)$. From the 2-dimensional worldsheet perspective the fields X^m are bosons while G and B are field dependent couplings. From the spacetime perspective, X^m are the co-ordinates of spacetime parametrized by the worldsheet co-ordinates. We choose co-ordinates $\{X^1, X^i\}$ for $i > 1$ in such a way that the symmetry acts by shifting along the X^1-direction only. The X^1-direction is special because it is compact and it is the direction in which our isometry acts. Then we may write

$$S' = \int d^2z[G_{11}A\bar{A} + L_{1i}A\bar{\partial}X^i + L_{i1}\partial X^i\bar{A} + L_{ij}\partial X^i\bar{\partial}X^j + \tilde{X}^1(\partial\bar{A} - \bar{\partial}A)]$$

where $L_{mn} = G_{mn} + B_{mn}$, and the background fields are all independent of X^1. Note the replacement

$$(\partial X^1, \bar{\partial} X^1) \to (A, \bar{A}) \tag{1.4}$$

where $(A, \bar{A})$ forms the auxiliary [10] gauge field on the worldsheet. We may view this replacement as gauging the shift symmetry of the original σ-model by a minimal coupling to the gauge field A [10]:

$$\partial X^i \to DX^i = \partial X^i + A$$

[9]Isometries occur whenever the action $S(G, B)$ is invariant under spacetime diffeomorphisms in a direction which has a global symmetry from the 2-dimensional worldsheet perspective.

[10]An auxiliary field does not contain any dynamics and therefore does not contribute to the degrees of freedom present in the theory. For ordinary fields, when we solve the equations of motion, the degrees of freedom are the labels which parametrize our solution. That is, they are the initial conditions. However, when solving the equations of motion for the auxiliary field A, we find algebraic equations, which are not integrated, and therefore there are no initial conditions which need to be specified.

for $i > 1$. The last term in S' contains the Lagrange multiplier $\tilde{X}^1$, whose purpose is to impose the constraint $F = dA = 0$. The result is a gauged non-linear σ-model $S(g, b; A, \tilde{X}^1)$. To return to the original model, the Lagrange multiplier $\tilde{X}^1$ must be integrated out, which takes us back to the original theory $S(G, B)$. The result being the reversal of the arrow in (1.4), and recovering the initial σ-model. However, integrating out A using the gauge field equations:

$$\begin{aligned} A =& G_{11}^{-1}(\partial\tilde{X}^1 - L_i^1\partial X^i), \\ \bar{A} =& -G_{11}^{-1}(\bar{\partial}\tilde{X}^1 - L_i^1\bar{\partial}X^i), \end{aligned}$$

returns a different theory. These solutions tell you that A and $\bar{A}$ are determined as soon as you specify the X's. It is the σ-model $S(G, B)$ written in terms of new background fields [11]. This is the *dual* theory with action

$$S'' = \int d^2z[\tilde{g}_{mn}(X) + \tilde{b}_{mn}(X)]\partial y^m\bar{\partial}y^n$$

written in terms of the co-ordinates $y^m = \{\tilde{X}^1, X^i\}$. The Lagrange multiplier represents the dual co-ordinate, the direction for which the dual theory is isometric. The dual fields are related to the original fields, as derived by Buscher

$$\begin{aligned} \tilde{g}_{11} =& (G_{11})^{-1}, \quad \tilde{g}_{1i} = (G_{11})^{-1}B_{1i}, \quad \tilde{b}_{1i} = -(G_{11})^{-1}G_{1i} \\ \tilde{g}_{ij} =& G_{ij} - (G_{11})^{-1}(G_{i1}G_{1j} + B_{i1}B_{1j}) \\ \tilde{b}_{ij} =& B_{ij} - (G_{11})^{-1}(G_{i1}B_{1j} + B_{i1}G_{1j}). \end{aligned} \tag{1.5}$$

This method works in the case of abelian isometries [12] [56]. Through recent developments, many derivations of the transformation rules for Ramond-Ramond fields [52,63–67] exist. We will first consider such rules in the context of T-duality described by an $O(D, D)$ symmetry group.

[11] This can also be done using a Hamiltonian approach through canonical transformations [54].

[12] Global issues were first raised in [59] and are subsequently discussed in [61] and [62].

A Note on Open String T-duality[13]

As we have seen, in the presence of a compact dimension, closed strings exhibit fundamentally new states due to their extended nature: that is, closed strings are able to wrap around the compact dimension. This winding makes it impossible to contract closed strings to a point. Since open string endpoints on a space-filling $D25$-brane are allowed to move freely, open strings can indeed always be contracted to a point. Compared to closed strings, open strings do not exhibit any fundamentally new states in the presence of a compact dimension. Furthermore, the string spectrum for an open string on a space-filling $D25$-brane with radius of compactification R and a related open string on a $D25$-brane with radius of compactification $\tilde{R} = R/\alpha'$, do not coincide. This implies that open strings on $D25$-branes are able to distinguish between the two different compactifications. To preserve T-duality in the presence of open strings, we find that T-duality relates a spacetime with radius R and a $D25$-brane to a spacetime with radius $\tilde{R} = R/\alpha'$ and a $D24$-brane. In the dual space, X^1 is the Dirichlet direction of the $D24$-brane. We set $X^1 = 0$ to be the position of the brane along the compact direction. All open string endpoints are then held attached to points with $X^1 = 0$ in the dual direction. Because the string endpoints are kept fixed, new open string configurations exist that prevent open strings from being contracted away.

Therefore, an open string stretching from $X^1 = 0$ to $X^1 = 2\pi R$ winds around the compact direction once. Open strings may wind around the compact direction any number of times just like closed strings, moreover, open strings may even resemble closed strings, however they are not closed since the open string endpoints do not need to coincide.

1.2.2 Generalised Geometry

The fundamental theory of relativity tells us that physics is the same for all observers, that is, physics is co-ordinate invariant. This idea is best formulated mathematically using differential geometry. One considers a tangent bundle [14] to the given spacetime manifold, equipped with a metric tensor. It is the metric tensor which acts as the dynamical object in the theory, 'encoding' the physical content. The Riemannian metric of general relativity provides a measure of distance in this theory. But what about theories requiring a metric and an antisymmetric 2-form to define their background? The answer is, we expect that these two fields should be treated equally and therefore combined to form the dynamical object. This is done by developing an associated geometrical theory, using the $O(D, D)$ structure group. This is the *generalized tangent bundle*

[13] This information box is an excerpt from [?], Chapter 18.

[14] A tangent bundle of a differential manifold M is a manifold TM , which assembles all the tangent vectors in M .

described in the context of *generalized geometry* [69–72]. The dynamical object is now given by the *generalized metric*, M :

$$M = \begin{bmatrix} G - BG^{-1}B & -BG^{-1} \\ G^{-1}B & G^{-1} \end{bmatrix}.$$

The generalized *dilaton*, d, is

$$e^{-2d} = e^{-2\Phi}\sqrt{\det G}.$$

The generalized metric provides the most efficient description of the action of T-duality on the background. In this set-up, T-duality is generated by $O(D, D)$ transformations with the following action

$$M \to M' = \mathcal{T}^{-1} M \mathcal{T}$$

with d remaining invariant. Then, T-duality is implemented using the following matrix [73]

$$\text{T-duality in the } k^{th} \text{ direction}: \quad \mathcal{T}_{T(k)} = \begin{bmatrix} \mathbf{1} - 1_k & 1_k \\ 1_k & \mathbf{1} - 1_k \end{bmatrix},$$

where 1_k is the $D \times D$-matrix with 1 as the k^{th} diagonal entry. This vector ensures we are dualising along the chosen direction, the k^{th} direction. Dualising along several co-ordinate directions generalizes naturally to $\mathcal{T}_{v1}\mathcal{T}_{v2} = \mathcal{T}_{v1+v2}$. The dual metric G' is then read off from the '(G^{-1})' term which corresponds to the matrix in the 4th quadrant of M'. The change in the dilaton is given by

$$\Delta\Phi = \frac{1}{4}\log\frac{\det G'}{\det G}.$$

T-duality in Field Theory

T-duality relies on the extended nature of strings to wrap around compact dimensions. The existence of winding modes resulting from such wrappings, and also the momentum modes underlie T-duality. The result of the action of T-duality being the creation of a connection between the physics (of strings) defined on geometrically distinct backgrounds. Double Field Theory[15] [77–79] tries to incorporate T-duality as a symmetry of field theory. At first the idea does seem bizarre, given that T-duality is a unique symmetry of strings, and that field theory describes 0-dimensional particles which are unable to wrap around compact dimensions. Therefore it is quite natural to assume that if we were to incorporate a T-duality symmetry in field theory, we should take into account information about the winding modes, somehow. To do this, we need to assign more degrees of freedom to our particles, accounting for their inability to reproduce stringy-like winding properties [74]. Double Field Theory incorporates these new degrees of freedom by doubling the space of co-ordinates. Thus ordinary and winding co-ordinates are considered to be the co-ordinates of a doubled manifold[16].

[15] For a review, consult [74–76].

[16] For any manifold M with a boundary ∂M , we may define its double to be the manifold obtained by gluing two copies of M together along their common boundary [80]. This does not mean that the double is always closed. In fact, this concept makes sense for any manifold. However, when discussing doubles, one is primarily referring to manifolds M which are compact (and thus closed) with non-empty boundaries ∂M .

Effects on Ramond-Ramond Fields

Hassan [53], and later Fukuma et al [81] described the changes made to the RR fields due to the action of T-duality by expressing the fields as a set of fermionic (or grassmannian) creation and annihilation operators, ψ^m and ψ_m respectively. These are combined to arrive at the potentials $C^{(p)}$. They are then given by

$$|C\rangle = \sum_p \frac{1}{p!} C^{(p)}_{mn...p} \psi^m \psi^n ... \psi^p |0\rangle,$$

with corresponding algebra

$$\{\psi_m, \psi^n\} = \delta^n_m, \quad \{\psi^m, \psi^n\} = \{\psi_m, \psi_n\} = 0.$$

The annihilation operators ψ_m annihilate the vacuum $|0\rangle$. T-duality acting along the m^{th} direction may be expressed as having the following operator

$$T_m = \psi^m + \psi_m. \tag{1.6}$$

acting on the vacuum $|0\rangle$. Note that different T_m anti-commute, so the sign of the final $C^{(p)}$ after T-dualising depends on the ordering, although this has no physical effect. The following subtleties apply

1. When considering a non-trivial dilaton, we ought to apply the above rules to $\mathrm{e}^{\Phi} dC^{(p)}$ instead [22, 23]. This is also mentioned in the last paragraph of [81].
2. A T-duality along the time direction exchanges real and imaginary fluxes [83], therefore (1.6) applies for $m \neq 0$ and we define $T_0 = i(\psi^0 + \psi_0)$.
3. For the case when the B-field is non-zero, these operators apply to the modified potential D, introduced in [81], rather than the potential C.

We may express the effects on the RR-fields in another way, as Hassan did in [52]. Here we may act directly on the fields $F^{(p)}$. Then performing a duality in the x^9-direction, for example, returns new fluxes

$$F^{(p)'}_{9no...q} = F^{(p-1)}_{no...q} - \frac{p-1}{G_{99}} G_{9[n} F^{(p-1)}_{9o...q]},$$
$$F^{(p)'}_{mn...q} = F^{(p+1)}_{9mn...q} - pB_{9[n} F^{(p)'}_{o...q]}$$

Since all our cases deal with diagonal G and a B which is zero, the second terms in each equation above vanish. This approach also requires one to remember to include a factor of i when dualising along time [83]. Appropriate factors also need to be inserted when the dilaton is non-trivial.

1.3 Non-Abelian T-duality

As we have already discussed, T-duality is an equivalence between string theories propa- gating on two different spacetime geometries which contain some isometries [19]. Simply put, it describes strings on a circle of radius R and those strings on a circle of radius α'/R as equivalent. In general, a map, given by the Buscher rules, is provided by the action of T-duality. This map relates one solution of supergravity to another. The Buscher procedure can naturally be generalized to include target spaces equipped with a non-Abelian isometry group H. For such a case, the Buscher procedure is followed almost exactly, except that the gauge fields [17] now take values in the algebra of H. This also produces a map from one solution of supergravity to another, as in the Abelian case. This acts as a solution generating technique for supergravity theories.

This natural generalisation was proposed in [84] (see [85] for earlier work). Although some attention was paid to this new duality in [86, 87], progress was impeded until it was worked out how to embed this duality symmetry in supergravity. This meant knowing the full transformation, including that of the RR fields, which was later found by Sfetsos and Thompson in [88]. Presently, we are at the stage where much more work and development has been done and progress has been made toward our understanding of non-Abelian T-duality [19,19,90–93], though there are still many things which remain a mystery. For example, non-Abelian T-duality might be extended to an exact duality of the full superstring theory, but at the moment this is something which remains un- clear. Although it appears that the dualisation procedure for Abelian and non-Abelian T-dualities are very similar, we list two important differences:

1. The isometry of the initial target space geometry is destroyed by non-Abelian T- duality, even if only partially. Furthermore, the supersymmetry of the model may be potentially destroyed too [18].

2. Global issues arise as a result of applying the Buscher procedure to worldsheets of higher genera, that is, beyond tree level. Therefore, non-Abelian T-duality is not an exact (or full) symmetry of string perturbation theory, existing only as a tree-level symmetry.

Earlier work on non-Abelian T-duality for cases involving purely Neveu-Schwarz backgrounds can be found in [84,86,87,94–97]. After the publication of [88], there was a resurgence of interest in the subject of non-Abelian T-duality, leading to the extension of the Buscher procedure to geometries containing RR fields in addition to NSNS fields. Furthermore, non-Abelian T-duality was originally studied in the context of backgrounds possessing $SU(2)$ isometries. Then, performing non-Abelian T-duality on the $SU(2)$ isometry group within the sphere of type IIB

[17] The gauge field is a 1-form with values in the Lie algebra associated to the gauged isometry group [56].

[18] It is possible to recover this lost isometry as a non-local symmetry of the σ-model. The corresponding σ-models are canonically equivalent [84].

$AdS_5 \times S^5$ causes something interesting to occur. The action of T-duality produces a geometry which is a solution of type IIA supergravity [88]. Consider the background containing an $SO(4) = SU(2)_L SU(2)_R$ isometry, then the round S^3 metric is written as

$$ds^2 = d\theta^2 + d\phi^2 + 2\cos\theta d\phi d\psi + d\psi^2.$$

Next, perform the T-duality with respect to one of the $SU(2)$ isometries, say the left [19], then we will find a geometry that interpolates between $\mathbb{R} \times S^2$ and $\mathbb{R}^3$ with the following metric

$$\hat{ds}^2 = dr^2 + \frac{r^2}{1+r^2}(d\theta^2 + \sin^2\theta d\phi^2).$$

However, there is also an NSNS antisymmetric 2-form field and scalar dilaton present for the dual geometry:

$$\hat{B} = \frac{r^3}{1+r^2} vol(S^2), \quad \hat{\Phi} = -\frac{1}{2}\ln(1+r^2).$$

We are also interested in how the RR fields transform. Suppose that the initial geometry considered above is supported by a RR 3-form

$$F_3 = vol(S^3).$$

Dual fluxes are extracted using the following equation

$$e^{\hat{\Phi}} \hat{\not{F}} = \not{F}.\Omega^{-1},$$

where the slashed terms represent the RR poly-form made up of the sum of the RR forms contracted with γ-matrices: $\not{F}_p = \frac{1}{p!}\Gamma_{\mu_1 \dots \mu_p} F^{\mu_1 \dots \mu_p}$, to form a bispinor. The matrix Ω's inverse is key to the transformation procedure and given by

$$\Omega^{-1} = \frac{1}{\sqrt{1+r^2}}(-\Gamma^{123} + r\Gamma^r).$$

From this we may determine that the dual geometry contains a 0-form and a 2-form:

$$F_0 = 1, \quad F_2 = \frac{r^3}{1+r^2} vol(S^2).$$

where the 0-form, F_0 , is called the *Romans mass* [20] [19]. When this dual geometry is embedded into a type IIA supergravity background, a solution of massive IIA supergravity is found. A massive IIA supergravity results whenever the Romans mass is non-zero. Notice that the type of string theory has changed from type IIB to IIA. The fact that the isometry group being dualised has an odd dimension means that the supergravity will change in type. For even-dimensional isometry groups, the supergravity type remains invariant.

[19] Note that if, instead, we performed T-duality with respect to the right $SU(2)$, the metric and background would remain unchanged. This is because we are dealing with type IIA string theory, which is parity invariant, and all that relates the two $SU(2)$'s is a parity transformation.

[20] The Romans mass was originally introduced as a 'cosmological constant' by Romans in 1986. However it was later understood to be an RR flux to be treated as any of the other RR fluxes, all of which are mixed by T-duality.

1.3.1 Some Technical Details

As mentioned earlier, the most well-studied version of non-Abelian T-duality exists for backgrounds having an $SU(2)$ isometry. Considering such a background we may write the metric as [88]

$$ds^2 = G_{\mu\nu}(x)dx^\mu dx^\nu + 2G_{\mu i}(x)dx^\mu L^i + \frac{1}{2}g_{ij}(x)L^i L^j$$

where $\mu = 1, 2, ..., 7$ and $i, j = 0, 8, 9$. The L^i's are the $SU(2)$ *Maurer-Cartan* forms [21]

$$L^i_\pm = -i\text{Tr}(t^i g^{-1}\partial_\pm g).$$

where $t^i = \frac{1}{\sqrt{2}}\sigma^i$ are the $SU(2)$ generators. The group element is given by

$$g = e^{\frac{i}{2}\phi\sigma_3} \cdot e^{\frac{i}{2}\theta\sigma_2} \cdot e^{\frac{i}{2}\psi\sigma_3},$$

with $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$. It is assumed that the NS sector also comprises an antisymmetric 2-form

$$B = B_{\mu\nu}(x)dx^\mu \wedge dx^\nu + B_{\mu i}(x)dx^\mu \wedge L^i + \frac{1}{2}b_{ij}(x)L^i \wedge L^j$$

and dilaton

$$\Phi = \Phi(x).$$

Note that all the dependence on the $SU(2)$ Euler angles θ, ϕ and ψ is contained in the Maurer-Cartan 1-forms. All remaining data is dependent on the fields x^μ, called *spectator fields.*

NS Sector

The Lagrangian density for the NS sector fields is given by

$$\mathcal{L} = Q_{\mu\nu}\partial_+ X^\mu \partial_- X^\nu + Q_{\mu i}\partial_+ X^\mu L^i_- + Q_{i\mu}L^i_+\partial_- X^\mu + E_{ij}L^i_+ L^j_-,$$

where

$$Q_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}, \quad Q_{\mu i} = G_{\mu i} + B_{\mu i}, \quad Q_{i\mu} = G_{i\mu} + B_{i\mu}, \quad E_{ij} = g_{ij} + b_{ij},$$

and $\mu = 1, 2, ..., 7$ and $i, j = 0, 8, 9$. Non-Abelian T-duality is performed through the replacement of ordinary derivatives with covariant derivatives, as follows

$$\partial_\pm g \to D_\pm g = \partial_\pm g - A_\pm g,$$

followed by adding a Lagrange multiplier term to ensure that we have a flat connection (which further ensures that there is no curvature)

$$-i\text{Tr}(vF_\pm), \quad F_\pm = \partial_+ A_- - \partial_- A_+ - [A_+, A_-].$$

[21] For group theory conventions, see Appendix A.

where the v_i's are Lagrange multipliers. There is more than one Lagrange multiplier present when considering a non-Abelian theory. This is because instead of having a single field, we now have a matrix of fields. Under this transformation, the spectator fields, x^μ remain inert. Partial integration results in the Lagrange multiplier taking the form

$$\mathrm{Tr}(i\partial_+ vA_- - i\partial_- vA_+ - A_+ fA_-), \quad f_{ij} = f_{ij}^k v_k.$$

At this stage, we have managed to introduce three new degrees of freedom. To eliminate three of the variables, we need to fix a gauge. The simplest choice available to us is $g = \mathbf{1}$. Then integrate out the gauge fields to get

$$A_+^i = iM_{ij}^{-1}(\partial_+ v_j + Q_{\mu j}\partial_+ X_\mu), \quad A_-^i = -iM_{ij}^{-1}(\partial_- v_j - Q_{j\mu}\partial_X^\mu), \tag{1.7}$$

where the matrix M is given by

$$M_{ij} = E_{ij} + f_{ij}.$$

Then substituting (1.7) back into the original Lagrangian yields the *dual* Lagrangian

$$\hat{\mathcal{L}} = Q_{\mu\nu}\partial_+ X^\mu \partial_- X^\nu + (\partial_+ v_i + \partial_+ X^\mu Q_{\mu i})M_{ij}^{-1}(\partial_- v_j - Q_{j\mu}\partial_- X^\mu). \tag{1.8}$$

The v_i's now take on the role of dual co-ordinates, just like the Lagrange multiplier in the Abelian case. The background fields of the NS sector for the T-dual theory may be read off from (1.8) as

$$\begin{aligned} \hat{Q}_{\mu\nu} =& Q_{\mu\nu} - Q_{\mu i}M_{ij}^{-1}Q_{j\nu}, \quad \hat{E}_{ij} = M_{ij}^{-1}, \\ \hat{Q}_{\mu i} =& Q_{\mu j}M_{ji}^{-1}, \quad \hat{Q}_{i\mu} = -M_{ij}^{-1}Q_{j\mu}. \end{aligned}$$

For the case of Abelian T-duality, the dilaton receives a contribution at the quantum level. The same applies to non-Abelian T-duality with the following contribution

$$\hat{\Phi}(x, v) = \Phi(x) - \frac{1}{2}\ln(\det M).$$

Note that it is the inverse of the matrix M above which determines the dual geometry.

RR Sector

The transformation rules for the RR fields were first discovered in the context of Abelian T-duality in [63] for supergravity theories in 9 spatial dimensions [88]. These rules were arrived at by considering the action of T-duality on spinors. These rules were written down for the spacetime perspective in [52, 98], and for the Green-Schwarz string in [64, 65]. The pure spinor formalism was written down in [67, 99]. The current method [100] combines the RR fields with their Hodge duals [22] to form the bispinor as in equations (1.2) and (1.3). Then the non-Abelian T-dual theory is obtained by multiplication with Ω^{-1} . When transforming a type IIB

[22] See Appendix A for conventions.

supergravity into a massive type IIA supergravity, we may obtain the rules for the transformation of the RR fluxes by comparing both sides of [88]

$$\hat{P} = P.\Omega^{-1}.$$

For transformations from massive IIA to IIB, one simply switches the roles of P and $\hat{P}$.

1.4 Summary and Conclusion

Dualities provide a unifying link between the different string theories. In particular bosonic T-duality relates type IIA supergravity to type IIB supergravity. In its sim- plest form, we have shown that bosonic T-duality is an equivalence between a theory with a large spacetime radius and a theory with a small spacetime radius. A powerful approach used to derive the Buscher rules, which form a recipe for performing bosonic T-duality, is the path integral approach. This procedure comprises three main steps:

1. Begin with the string σ-model describing some spacetime and gauge a $U(1)$ isometry of this spacetime (or $SU(2)$ in the case of non-Abelian T-duality).
2. Using a flat connection for this gauge field, we insert a Lagrange multiplier.
3. Integrate by parts to obtain an action with a non-dynamical gauge field that may be eliminated by its equations of motion, to produce the T-dual σ-model.

In summary, given a non-linear σ-model with a target space that admits an isometry, the Buscher procedure gives us a well defined recipe for obtaining a T-dual σ-model. There are some subtleties that occur when trying to prove that the Buscher procedure holds on worldsheets of higher genera [10, 59]. Abelian T-duality Buscher rules are only valid to the lowest order in string perturbation theory, as is true for the non-Abelian case. However, by gauging an Abelian isometry, the duality may be extended to higher genus worldsheets. In this case, the shift symmetry has to be along a compact coordinate [59, 101]. The full set of bosonic T-duality transformations is given by the non-Abelian group $SO(D, D, Z)$. It is generated by T-dualities on all D circles, linear redefinitions of axes and discrete shifts of the Kalb-Ramond field B [49]. Naturally, we may wonder about the role that non-Abelian T-duality plays in the gauge/string correspondence. To answer this question, Sfetsos and Thompson [88] studied the dualisation of $AdS_5 \times S^5$ with an $SU(2)$ isometry group. Non-Abelian T-duality studied in this context has many efforts, which may be found in [19, 90, 102–104] and references therein. A far more thorough review of non-Abelian T-duality can be found in [99]. It is important to stress that Abelian T-duality is an *exact* duality of superstring theory whereas it is not known whether non-Abelian T-duality *is* a duality. However, it has proven useful as a solution generating technique, which acts at the level of the supergravity. The take home idea is that bosonic T-duality may be used to generate new solutions of supergravity. This is very useful for mapping very distinct geometries to one another.

Chapter 2

Introductory Fermionic T-Duality

2.1 Introduction

Fermionic T-duality [21,22] is the cousin of bosonic T-duality which extends the idea to a duality that acts on the whole superspace. Initially, the motivation to study fermionic generalisations to bosonic T-duality arose as a result of the need to understand dual superconformal symmetry with regards to scattering amplitudes of the string, as well as trying to understand its connection to integrability. Dual superconformal symmetry eventually helped show that $AdS_5 \times S^5$ was self-dual [21, 22]. This discovery shed light on hidden symmetries in the scattering amplitudes of supersymmetric theories like $\mathcal{N} = 4$ super Yang-Mills [26–28], and suddenly there was a great desire to explain these symmetries which fuelled research centred on the topic of fermionic T-duality. Let us take a step back to recall some of the things that we already know, for context. We have seen that bosonic T-duality has been crucial in linking the two type II string theories, and that the prerequisite for this duality transformation is an isometry on the background. For bosonic T-duality, Buschers procedure tells us to use a shift symmetry of the target space coordinate, corresponding to the isometry on the target space of the σ-model. Then, we are able to make some field redefinitions in the σ-model. Classically, the new σ-model is equivalent to the original σ-model, and both models have identical forms of the action. However, the new σ-models couplings [101], that is the metric and 2-form NSNS field look different. If the dilaton transforms as well, then the σ-models are equivalent at the quantum level. Additionally, in the bosonic case, the transformed background fields are related to the original fields by the *Killing vectors* corresponding to the isometry.

The generalisation to target spaces possessing *fermionic isometries*, or *supersymmetries*, as well as isometries, resulted in the duality of tree-level type II string theory called *fermionic T-duality* [101]. Under this duality only the RR fields and dilaton transform, leaving the rest of the NSNS sector (metric and NSNS 2-form) invariant. Analogous to bosonic T-duality, the transformation of the background supergravity fields are generated by the *Killing spinors* corresponding to the superisometry of the superspace. It was Berkovits and Maldacena [22] and simultaneously Beisert et. al. [21], who in 2008, managed to generalise the Buscher procedure for worldsheets with actions invariant under constant shifts of the spacetime fermionic coordinates θ^J, $J = 1, ..., n$, where n is the number of supersymmetries. The metric and NSNS 2-form remain invariant and the transformation laws of the RR fields are given in terms of the bispinor field strength. In type IIA string theory, the bispinor field strength is

$$F^{\alpha\beta} = \frac{1}{2} F^{(2)}_{a_1 a_2} (\Gamma^{a_1 a_2})^{\alpha\beta} + \frac{1}{4!} F^{(4)}_{a_1 \ldots a_4} (\Gamma^{a_1 \ldots a_4})^{\alpha\beta},$$

where the 2-form $F^(2)$ and 4-form $F^(4)$ are the RR field strengths. In type IIB string theory the bispinor field strength is[1]

$$F^{\alpha\beta} = F^{(1)}_a (\Gamma^a \sigma^1) + \frac{1}{3} F^{(3)}_{a_1 \ldots a_3} (\Gamma^{a_1 \ldots a_3} (i\sigma^2))^{\alpha\beta} + \frac{1}{2 \cdot 5!} F^{(5)}_{a_1 \ldots a_5} (\Gamma^{a_1 \ldots a_5} \sigma^1)^{\alpha\beta},$$

where the 1-form $F^(1)$, the 3-form $F^(3)$ and the 5-form $F^(5)$ are the RR field strengths. The 5-form $F^(5)$ is also self-dual with respect to the Hodge star operator[2]. There are important differences between bosonic T-duality and its fermionic cousin. We will note them here:

1. Fermionic T-duality is not an exact duality symmetry of string theory. The symmetry is broken at one loop in the string coupling, g_s.

2. The transformation laws for the background fields resulting from the fermionic Buscher procedure are quite different when compared to the bosonic case. Strikingly, the NSNS sector is left untouched save the dilaton, which picks up the following additive correction

$$\phi' = \phi + \frac{1}{2} \log C,$$

   where C is determined by the Killing spinors $(\epsilon, \hat{\epsilon})$, which parametrize the fermionic isometries. Note that this pair of Killing spinors represents a single supersymmetry.

3. The transformation of the bosonic fields in the RR sector may be written concisely in terms of the bispinor $F^{\alpha\beta}$:

$$e^{\phi'} F'^{\alpha\beta} = e^{\phi} F^{\alpha\beta} + 16 i \epsilon^{\alpha} \hat{\epsilon}^{\beta} C^{-1}, \tag{2.1}$$

   where the bispinor[3] is found, as before, by contracting all the RR fields in the theory with

[1] Notice the Pauli matrices present in the IIB case. Type IIB string theory is chiral (left/right asymmetric). As a result we are dealing with two-component Majorana-Weyl spinors $\eta = (\epsilon, \hat{\epsilon})$. Each of the Killing spinors in the pair have 16-components. Given that we are discussing type IIB string theory the two 16-dimensional Killing spinors have the same chirality.

[2] See appendix A for conventions.

[3] Note that the conventions for the factor in front of the second term on the right hand side of (2.1) differ from author to author.

the appropriate antisymmetrized products of Γ-matrices.

4. There is an important new feature which occurs for fermionic transformations. The Killing spinors need to be complex in order to satisfy the commutativity condition, which will be discussed below, this ensures that the symmetry we gauge in the Buscher procedure is Abelian. Therefore, our supersymmetries commute. The consequence is that performing a fermionic T-duality transformation will result in a complexified solution of supergravity.

5. It is desirable to return to a real background, as we would like to describe physical processes, to this end it is essential to apply a bosonic T-duality along a time-like Killing vector. The returns a real solution with real fluxes. However, this does mean that our solutions obtained by using fermionic T-duality as a 'solution generating technique' are limited to supergravity solutions possessing a time-like Killing vector.

Take note that the transformation law for the dilaton is very similar to the way that the dilaton changes under a bosonic T-duality with the essential difference being that in the fermionic case, the sign of the logarithm has changed. This difference turns out to be extremely important for establishing the exact T-self-duality of the $AdS_5 \times S^5$ background. We will deal with self-duality later but, simply put, it means that the original background is returned to after a sequence of bosonic and fermionic T-duality transformations has been applied. When applying bosonic transformations in order to establish self-duality, we always need to apply an even number. This occurs because the bosonic transformations swap the type IIA and type IIB backgrounds. Thus, to return to the same background, an even number of bosonic T-dualities is required.

2.2 The Fermionic Buscher Procedure

As before, we start by considering a σ-model, in this case a *Green-Schwarz* σ-model on the whole superspace, depending on both bosonic and fermionic worldsheet variables (x^m, θ^μ). Next, choose one of the fermionic directions, say θ^1, such that the worldsheet action is invariant under a constant shift of the fermionic variable. This means that

$$\theta^1 \to \theta^1 + \rho, \quad x^m \to x^m, \quad \theta^{\tilde{\mu}} \to \theta^{\tilde{\mu}}, \tag{2.2}$$

where ρ is a fermionic constant and $\tilde{\mu}$ runs over all fermionic degrees of freedom except $\mu = 1$. These types of backgrounds inherently preserve a supersymmetry [22]. Invariance under (2.2) implies that θ^1 will only appear as derivatives in the action, which is given by

$$S = \int d^2z \left[B_{11}(Y)\partial\theta^1\bar{\partial}\theta^1 + L_{1M}(Y)\partial\theta^1\bar{\partial}Y^M + L_{M1}(Y)\partial Y^M\bar{\partial}\theta^1 + L_{MN}\partial Y^M\bar{\partial}Y^N \right],$$

where $Y^M = (x^m, \theta^{\tilde{\mu}})$ and $M = (m, \tilde{\mu})$ ranges over all indices except for $\mu = 1$. We define $L_{MN}(Y) = G_{MN}(Y) + B_{MN}(Y)$ where G_{MN} is the graded-symmetric tensor and B_{MN} is the graded-antisymmetric tensor defined by

$$G_{MN} = (-1)^{MN} G_{NM}, \quad B_{MN} = -(-1)^{MN} B_{MN},$$

with

$$(-1)^{MN} = \begin{cases} -1 & \text{M, N both fermionic,} \\ +1 & \text{Otherwise} \end{cases}$$

where the background fields are contained as components. Once again, we introduce a vector field $(A, \bar{A})$, which is composed of fermionic gauge fields. The derivatives are then replaced with the vector fields as follows:

$$(\partial\theta^1, \bar{\partial}\theta^1) \to (A, \bar{A}),$$

where $(A, \bar{A})$ forms an auxiliary field on the worldsheet. This replacement is viewed as gauging the shift symmetry of the original Green-Schwarz σ-model by a minimal coupling to the fermionic gauge field. If $B_{11}(Y)$ is non-zero then we may apply the Buscher procedure in order to T-dualize the σ-model with respect to θ^1. Additionally, a Lagrange multiplier term $\tilde{\theta}^1$ is added to the action for the purpose of imposing that the vector field is a derivative of a fermionic scalar. The resulting action is

$$S = \int d^2z \left[B_{11}(Y) A\bar{A} + L_{1M}(Y) A\bar{\partial}Y^M + L_{M1}(Y)\partial Y^M \bar{A} + L_{MN}\partial Y^M \bar{\partial} Y^N + \tilde{\theta}^1(\partial\bar{A} - \bar{\partial}A) \right].$$

To return to the original Green-Schwarz σ-model, the Lagrange multiplier $\tilde{\theta}^1$, must be integrated out. This imposes that $A = \partial\theta^1$ and $\bar{A} = \bar{\partial}\theta^1$. Alternatively, we may integrate out the fermionic gauge field instead, using the equations of motion for A given by

$$\begin{aligned} \partial\tilde{\theta}^1 &= B_{11}A + L_{M1}\partial Y^M, \\ \bar{\partial}\tilde{\theta}^1 &= B_{11}\bar{A} - (-1)^{s(M)} L_{1M}\bar{\partial}Y^M, \end{aligned}$$

where the exponent $s(M)$ is zero if M is a bosonic index and one if M is fermionic. This results in the T-dualized action

$$\tilde{S} = \int d^2z \left[B'_{11}(Y)\partial\tilde{\theta}^1\bar{\partial}\tilde{\theta}^1 + L'_{1M}(Y)\partial\tilde{\theta}^1\bar{\partial}Y^M + L'_{M1}(Y)\partial Y^M\bar{\partial}\tilde{\theta}^1 + L'_{MN}\partial Y^M\bar{\partial}Y^N \right]$$

where

$$B'_{11} = -\left(B_{11}\right)^{-1}, \quad L'_{1M} = \left(B_{11}\right)^{-1}L_{1M},$$
$$L'_{M1} = \left(B_{11}\right)^{-1}L_{M1}, \quad L'_{MN} = L_{MN} - \frac{1}{B_{11}}L_{1N}L_{M1}.$$

There is a change induced in the dilaton due to the measure factor coming from the integration over the fermionic vector field (see [106] to see how the measure factor arises for the case of integration over a bosonic field). The change is given by

$$\phi' = \phi + \frac{1}{2}\log B_{11}.$$

Note that the change in ϕ as a result of fermionic T-duality has a logarithm with the opposite sign from that appearing in the change in ϕ as a result of bosonic T-duality. This is because, whilst the integration of the vector field has the same formal structure as in the bosonic case, we are instead integrating over an anticommuting variable, which picks up an additional minus sign. Fermionic T-duality transformations leave the NSNS fields G_{MN} and B_{MN} invariant, as mentioned above. Therefore, fermionic T-duality is primarily a transformation acting on the RR sector. The transformation is written in terms of the bispinor field strength given in (2.1). Furthermore, the fermionic isometries are assumed to be Abelian [22], so that they satisfy the following *commutativity* constraint

$$\epsilon\gamma^m\epsilon + \hat{\epsilon}\gamma^m\hat{\epsilon} = 0, \tag{2.3}$$

in type IIB and

$$\bar{E}\Gamma^m E = 0 \tag{2.4}$$

in type IIA for a Majorana spinor $E = \epsilon + \hat{\epsilon}$. The two spinors ϵ and $\hat{\epsilon}$ are not independent, instead they are related by the Killing spinor equations and the above constraint. This constraint is unique to fermionic T-duality and cannot be solved for real spinors, therefore the Killing spinors need to be artificially complexified, as we will see in the next section. The consequence of this is that performing a fermionic T-duality transformation will result in complex RR fields. Once a pair of Killing spinors satisfying the commutativity condition have been found, the next step is to calculate the auxiliary scalar field C which is defined as follows

$$\begin{aligned}\partial_m C =& i\epsilon^{\alpha}(\gamma_m)_{\alpha\beta}\epsilon^{\beta} - i\hat{\epsilon}^{\hat{\alpha}}(\gamma_m)_{\hat{\alpha}\hat{\beta}}\hat{\epsilon}^{\hat{\beta}} \quad \text{for type IIB,}\\ \partial_m C =& i\bar{E}\Gamma_m\Gamma^{11}E \quad \text{for type IIA.}\end{aligned}$$

The auxiliary field C is the $\theta = \hat{\theta} = 0$ component of B_{11}. Using the constraint (2.3), we are able to somewhat simplify the type IIB expression to

$$\partial_m C = 2i\epsilon\gamma_m\epsilon. \tag{2.5}$$

The dilaton can be written in terms of this auxiliary field as follows

$$\phi' = \phi + \frac{1}{2}\log C, \tag{2.6}$$

and we recall that the RR fields transform according to

$$e^{\phi'}F' = e^{\phi}F + \frac{16}{i}\frac{\epsilon\otimes\hat{\epsilon}}{C}. \tag{2.7}$$

For the case when multiple fermionic T-dualities are performed, with respect to several supersymmetries parametrized by pairs of Killing spinors $\varepsilon_i = (\epsilon_i, \hat{\epsilon}_i)$ with $i \in 1, ..., n$, we may generalise equations (2.5), (2.6) and (2.7) to

$$\partial_m C_{ij} = 2i\epsilon_i\gamma_m\epsilon_j, \tag{2.8}$$

$$\phi' = \phi + \frac{1}{2}\sum_{i=1}^{n}(\log C)_{ii}, \tag{2.9}$$

$$e^{\phi'}F' = e^{\phi}F + \frac{16}{i}\sum_{i,j=1}^{n}(\epsilon_i\otimes\hat{\epsilon}_j)(C^{-1})_{ij}. \tag{2.10}$$

The commutativity condition, for all $i, j \in 1, .., n$, becomes

$$\epsilon_i \gamma_m \epsilon_j + \hat{\epsilon}_i \gamma_m \hat{\epsilon}_j = 0. \tag{2.11}$$

2.3 A Recipe for Fermionic T-duality

Bakhmatov, in his thesis [16], formulated a great recipe for performing fermionic T-duality on any solution. In this section, we will present this recipe in detail.

Performing Fermionic T-duality

The steps are:

1. Find the Killing spinors of the solution.
2. Choose a complex linear combination of the Killing spinors from the first step, ensuring that they satisfy the commutativity condition (2.3).
3. Calculate the auxiliary scalar field C, using (2.5).
4. If there are any RR fields existing in the original background, they should be substituted into
$$F^{\alpha\beta} = (\gamma^{\mu})^{\alpha\beta} F_{\mu} + \frac{1}{3}(\gamma^{\mu_1\mu_2\mu_3})^{\alpha\beta} F_{\mu_1\mu_2\mu_3} + \frac{1}{2.5!}(\gamma^{\mu_1\ldots\mu_5})^{\alpha\beta} F_{\mu_1\ldots\mu_5}. \tag{2.12}$$
5. One then uses $F^{\alpha\beta}$, ϵ^{α}, $\hat{\epsilon}^{\beta}$ and C to calculate the transformed RR background $F'^{\alpha\beta}$ using (2.7).
6. Use (2.12), but this time use it to find the contributions of F_1, F_3 and F_5 to the transformed background $F'^{\alpha\beta}$ separately.

A few comments are in order. Type IIA supergravity contains two 16-component, Majorana-Weyl spinors with opposite chirality. It is convenient to combine the two Majorana-Weyl spinors into a single 32-component spinor $E = (\epsilon, \hat{\epsilon})$, and to then work with constraint equations and the Killing spinor equations in terms of the full 32×32 Γ-matrices. The type IIB case is handled differently. Our Killing spinors are, again, $\varepsilon = (\epsilon, \hat{\epsilon})$, but now the two 16-component spinors have the same chirality. For IIB string theory, we work with constraint equations and the Killing spinor equations in terms of the 16×16 γ-matrices. The fermionic directions along which we dualise are found from the Majorana-Weyl spinors by creating complex linear combinations of the Killing spinors found in step 1. The next step is to use the complex Killing spinors to write down a differential equation (**??**), which may be integrated to solve for the auxiliary scalar field C. The bispinor (**??**) in step 4 contains antisymmetrized products of γ-matrices with the RR fields. The single γ-matrix and the product of five γ-matrices end up being symmetric overall. However, the triple product of γ-matrices is antisymmetric overall. After calculating the new bispinor $F'^{\alpha\beta}$ in step 5, we encounter the first non-trivial, challenging part of the transformation in step 6. Here we need to extract the transformed RR fields from the new bispinor. To find the corresponding 1-form, 3-form and 5-form, we need to separate the matrices occurring in (**??**) into symmetric and antisymmetric parts. This done by brute force using *Mathematica*. Steps 2, 4 and 5 are all automated with *Mathematica*, since all these equations involve calculations with many 32×32 (type IIA) and 16×16 (type IIB) matrices which are very cumbersome.

2.4 Self-duality of the $AdS_5 \times S^5$ Background

Before moving on to the study of backgrounds with less than maximal supersymmetry, it is instructive to review the maximally supersymmetric solution, $AdS_5 \times S^5$, first. This review follows the structure in [23].

To show that $AdS_5 \times S^5$ is self-dual under a sequence of bosonic and fermionic T-dualities, we choose to perform bosonic T-duality along the (x^0, x^1, x^2, x^3)-directions of AdS_5 in Poincaré coordinates. Following this, we perform fermionic T-duality along 8 Poincaré supersymmetries [21, 22]. The corresponding Poincaré Killing spinors are those spinors independent of the boundary coordinates of AdS_5. In Poincaré coordinates the metric on the AdS_5 space is

$$ds^2(AdS_5) = \frac{1}{r^2}(dx^m dx_m + dr^2),$$

where $m = 0, 1, 2, 3$ and r is the radial direction in AdS_5. Solving for the Poincaré Killing spinors η of $AdS_5 \times S^5$ gives the following solution

$$\eta = \frac{1}{\sqrt{r}}\tilde{\eta},$$

where we have suppressed the angular dependence of the S^5 in $\tilde{\eta}$. We will see this form of the solution appearing for the less supersymmetric backgrounds in Part II. The $AdS_5 \times S^5$ geometry has a self-dual[4] 5-form RR flux and a trivial (constant) dilaton. We may write the flux bispinor (which is used in the fermionic Buscher procedure) as [22]

$$e^{\phi} F^{\alpha\hat{\beta}} = (\gamma_{01234})^{\alpha\hat{\beta}},$$

[4]Self-dual with respect to the Hodge star operator defined in the Appendices.

where 0, 1, 2, 3 correspond to the x^m directions and 4 corresponds to the radius of AdS_5. The maximally supersymmetric geometry, $AdS_5 \times S^5$, has 32 supersymmetries which are reduced to 16 by restricting to Poincaré supersymmetries. The commutativity constraint 2.3 further reduces this number to 8 pairs of complexified Killing spinors corresponding to 8 commuting fermionic directions. The matrix C can be calculated using 2.5 where upon being inverted it is used to find the transformed bispinor 2.10. The T-dual bispinor is [21, 22]

$$e^{\tilde{\phi}} \tilde{F}^{\alpha\hat{\beta}} = (i\gamma_4)^{\alpha\hat{\beta}}.$$

Using C again allows us to find the shift in the dilaton

$$\tilde{\phi} = \phi + 4 \log r.$$

To recover the original geometry we still need to perform bosonic T-dualities along the chosen x^m. It is during this process that the difference in sign before the logarithmic term in the dilaton shifts due to bosonic and fermionic transformations plays a crucial role. Bosonic T-duality produces a shift $\Delta\phi = -4 \log r$ which undoes the shift caused by fermionic T-duality. Note, that at this stage we are still left with a complex geometry, meaning that we are not done. To arrive at the original background we need to perform a timelike T-duality which attaches an i to all the RR fields [83] and recovers a real solution.

2.5 Summary and Conclusion

The aim of this chapter was to give a brief introduction to the concept of fermionic T-duality. For a thorough review with many references see [23], as well as [107] for some concrete examples, illustrating the fermionic T-duality transformation.

In summary, fermionic T-duality is a cousin of bosonic T-duality which extends the idea to work on the entire superspace. It is a tree-level duality symmetry of string perturbation theory and it preserves the supersymmetry of the background, with the Killing spinors belonging to the Abelian subalgebra of the superalgebra [16]. From the supergravity perspective, fermionic T-duality rotates Killing spinors, preserving chirality in addition to supersymmetry. The pair $\varepsilon = (\epsilon, \hat{\epsilon})$ generates a single supersymmetry transformation.

To apply the Buscher procedure to the fermionic generalisation, we require that our fermionic isometries (or supersymmetries) commute [23]. The immediate consequence of this is that the fermionic T-duality transformation does not admit Majorana (or real) Killing spinors. Therefore, we need complex Killing spinors which result in complex solutions to supergravity.

In the introduction we noted the differences between bosonic and fermionic T-duality transformations. Most notably, the difference in the sign appearing in front of the logarithm term in the dilaton shift changes to 'plus' for the fermionic case, where it was 'minus' in the bosonic case. This change ends up being essential for establishing the self-duality of $AdS_5 \times S^5$. In fact, the shift of the dilaton due to bosonic T-duality along the AdS_5 is exactly cancelled by the dilaton shift caused by fermionic T-duality, which results in a symmetry at the quantum level.

For completeness we give the explicit form of the Killing spinors of the T-dual theory:

$$\tilde{\epsilon}_i^{\alpha} = (C^{-1})_{ij}\epsilon_j^{\alpha}, \quad \tilde{\hat{\epsilon}}_i^{\hat{\alpha}} = (C^{-1})_{ij}\hat{\epsilon}_j^{\hat{\alpha}}.$$

Finally, we note that fermionic T-duality transformations do not commute with bosonic T-duality transformations.

Part II

Supergravity

Chapter 3

Fermionic T-duality of $AdS_d \times S^d(\times S^d) \times M$

3.1 Introduction

In this chapter we consider the self-duality of some non-maximally supersymmetric backgrounds as a result of performing a series of bosonic and fermionic T-dualities. We have aimed to keep our computations as simple as possible by working directly with the fields at the level of the supergravity. This simple approach is a great way to develop intuition regarding the mechanism of T-duality transformations, by observing the changes in the RR fields explicitly. With the machinery laid bare, it is easy to notice the need for bosonic T-dualities along some torus directions, a fact which was nearly missed when working at the level of the coset[1]. Although we find it most convenient to work with the IIA string theory as in [121], a bosonic T-duality will take us back to a IIB background. Thus, our proofs are also valid in the extension to IIB. The goal of this chapter is to provide an elementary explanation of the duality of these backgrounds, at the expense of generality. The recipe given in section 2.3 is what we apply to every example covered in this chapter. However, subtleties arise when considering the exceptional cases (i.e. backgrounds possessing two spheres) and therefore additional discussion is required. At the end of each example we present a diagram that summarizes the technical details of that section in order to provide a 'big picture' view. T-duality in the general context is considered in Chapter 4 which also discusses some of the examples presented in this chapter. This chapter is based on research covered in [36].

[1]The non-linear σ-models that we use in this thesis may be described by cosets G/H, where G is the isometry group of the spacetime manifold and H is the isotropy group. See Chapter 4 for details and the Appendices for mathematical definitions.

3.2 Type IIA Supergravity

Since we are working with IIA supergravity it is necessary to give the basic layout of fermionic T-duality in this formalism. Here we have two Majorana-Weyl spinors of opposite chirality. We use a basis in which the chirality operator is given by $\Gamma_{11} = \sigma^3 \otimes 1$. Then we may write the spinors as follows

$$\begin{pmatrix} \epsilon^\beta \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \hat{\epsilon}_\beta \end{pmatrix}, \quad \beta = 1, ..., 16.$$

We may combine the Killing spinors into a single 32-dimensional $E = (\epsilon, \hat{\epsilon})$. Greek indices denote curved spatial directions, given by $\mu, \nu = t, x, z, \theta_+, ..., \psi_-, u$. Latin indices represent flat spacetime coordinates given by $m, n = 0, 1, ..., 9$. Spinor indices are given by α and β. Following [121], the Killing spinors must ensure that the gravitino and dilatino variations[2] vanish, which implies that

$$D_\mu E = -\frac{1}{8} S \Gamma_\mu E, \quad TE = 0 \tag{3.1}$$

where the covariant derivative is

$$D_\mu E = \partial_\mu E + \frac{1}{4} \omega_{\mu np} \Gamma^{np} E$$

and

$$S \equiv \not{F}^{(2)} \Gamma_{11} + \not{F}^{(4)}, \quad T \equiv \frac{i}{16} \Gamma_m S \Gamma^m = \frac{3i}{8} \not{F}^{(2)} \Gamma_{11} + \frac{i}{8} \not{F}^{(4)}.$$

The supergravity solutions that we are interested in have the fermionic fields set to zero. Variations under supersymmetry have the following form:

$$\delta(\text{boson}) \propto \text{fermion}, \quad \delta(\text{fermion}) \propto \text{boson}.$$

Immediately it follows that the bosonic field supersymmetry variation is $\delta(\text{boson}) = 0$. By a supersymmetric solution, we mean that

[2] See appendix A for more details.

$$\delta(\text{boson}) = 0 = \delta(\text{fermion})\,.$$

Therefore, the only condition required to have any supersymmetry preserved is that the fermionic field supersymmetry variation vanishes: $\delta(\text{fermion}) = 0$. The Killing spinor equations (3.1) do precisely this, they set these fermionic variations to zero and we are able to determine what supersymmetries are present. The Buscher procedure for the fermionic generalisation, as derived by Berkovits and Maldacena [22], is employed where the direction along which we dualise is specified by a complex Killing spinor. That is, a fermionic isometry or, supersymmetry. For example, $\mathcal{E} = E_1 + iE_2$ is one such combination where $E_i = (\epsilon_i; \hat{\epsilon} i)$ are solutions of (3.1). We are able to perform T-duality along multiple directions $\mathcal{E}_i$ at once, however, they must obey a commutativity condition given by

$$\bar{\mathcal{E}}_i \Gamma_\mu \mathcal{E}_j = 0 \;\; \forall \mu, i, j, \quad \text{where } \; \bar{\mathcal{E}} \equiv \mathcal{E}^T \Gamma^0. \tag{3.2}$$

Equation (3.2) is the statement that our isometry is Abelian[3], which is required to perform fermionic T-duality. This is because Berkovits and Maldacena followed Buscher's procedure and gauged an Abelian symmetry (as was done for the bosonic case). Condition (3.2) is not solved by any real spinors, thus we use complex Killing spinors. The next step is to find a matrix C from

$$\partial_\mu C_{ij} = i \bar{\mathcal{E}}_i \Gamma_\mu \Gamma_{11} \mathcal{E}_j, \tag{3.3}$$

which gives us the change in the dilaton

$$\Delta\Phi \equiv \Phi' - \Phi = \frac{1}{2} \log(\det C), \tag{3.4}$$

and the change in the RR forms

$$\Delta F \equiv e^{\Phi'} F'^{\alpha}_{\beta} - e^{\Phi} F^{\alpha}_{\beta} = \frac{16}{i} (C^{-1})_{ij} \epsilon^{\alpha}_i \hat{\epsilon}_{j\beta}. \tag{3.5}$$

Note that in (3.5) the lower-case spinors are complex, i.e. $\mathcal{E}_i = (\epsilon_i, \hat{\epsilon}_i)$. The bispinor F encoding the RR forms is given by

[3]Note that at the moment, there does not seem to be a meaningful reason for why this must be true. Perhaps future discussions will reveal a deeper understanding. What we can say is that, if our isometries do not commute, the situation is more complex.

$$F^{\alpha}_{\beta} = \frac{1}{2!} F^{(2)}_{mn} (\gamma^m)^{\alpha\gamma} (\gamma^n)_{\gamma\beta} + \frac{1}{4!} F^{(4)}_{mnpq} (\gamma^m \gamma^n \gamma^p \gamma^q)^{\alpha}_{\beta}. \tag{3.6}$$

The lower-case gamma matrices are related to the upper-case gamma matrices by

$$\Gamma^m = \begin{bmatrix} 0 & (\gamma^m)^{\alpha\beta} \\ (\gamma^m)_{\alpha\beta} & 0 \end{bmatrix}.$$

That is, the lower-case gamma matrices are the 16×16 off-diagonal blocks of the larger 32-dimensional gamma matrices. See Appendix A for our conventions.

3.3 $AdS_2 \times M$ Backgrounds

This section considers $AdS_2 \times S^2(\times S^2) \times M$ backgrounds, all of which possess 8 supersymmetries[4]. However, we neglect the superconformal Killing spinors in favour of the Poincaré Killing spinors as in [23]. Poincaré Killing spinors depend solely on the radial direction of the AdS subspace. These spinors are most useful in exhibiting self-duality and tend to have a simpler explicit form. Using Poincaré Killing spinors restricts us to considering 4 Killing spinors, which allows for 2 complex directions, upon complexification, along which to perform fermionic T-duality. The metric has a parameter α that has range $(0, 1)$, such that when $\alpha \to 1$, the second sphere approaches f flat spacetime. This decouples the sphere from the $AdS_2 \times S^2$ subspace which results in these directions being re-compactified into a T^2, contributing to torus directions required to make $AdS_2 \times S^2 \times T^6$. The general metric is given by

$$ds^2 = ds^2_{AdS} + \frac{1}{\alpha} ds^2_{S_+} + \frac{1}{1-\alpha} ds^2_{S_-} + \sum_i dy_i^2 = \eta_{mn} e^m_\mu e^n_\nu dx^\mu dx^\nu. \tag{3.7}$$

We adopt Poincaré coordinates for AdS_2

$$ds^2_{AdS_2} = \frac{-dt^2 + dz^2}{z^2},$$

with implied spin connection components $\omega_{t[01]} = \frac{1}{z}$. For the spheres we have the nested form

$$ds^2_{S^3_\pm} = d\theta^2_\pm + \sin^2\theta_\pm d\phi^2_\pm, \tag{3.8}$$

[4] The notation $AdS_2 \times S^2(\times S^2) \times T$ identifies two classes of solutions: the $AdS_2 \times S^2 \times T^6$ backgrounds and the exceptional $AdS_2 \times S^2(\times S^2) \times T^4$ backgrounds. Therefore, the brackets indicate that the second sphere is optional with the value of the parameter α specifying which background is being dealt with.

with implied spin connection components $\omega_{\phi_+[23]} = -\cos\theta_+$ and $\omega_{\phi_-[45]} = -\cos\theta_-$. Note that the spin connection components $\omega_{\mu ab}$ are independent of α although the vielbeins e_μ^m are not. For what follows we take $\Phi = 0$ and use the following RR flux:

$$\not{F}^{(4)} = \Gamma^{01}(-\Gamma^{67} + \Gamma^{89}) + \sqrt{\alpha}\Gamma^{23}(\Gamma^{68} + \Gamma^{79}) + \sqrt{1-\alpha}\Gamma^{45}(-\Gamma^{78} + \Gamma^{69}). \tag{3.9}$$

For $\alpha = 1$, bosonic T-duality along the x^5 direction takes us to the IIB $AdS_2 \times S^2 \times T^6$ case studied in [39], with $F^{(5)}$ only. Following [121] we may write

$$\not{F}^{(4)} = -4\Gamma^{0167}P_1(1 - P_2),$$

where

$$P_1 = \frac{1}{2}(1 + \Gamma^{6789}), \quad P_2 = \frac{1}{2}(1 + \sqrt{\alpha}\Gamma^{012378} + \sqrt{1-\alpha}\Gamma^{014568}),$$

are projection operators. The Killing spinor equation is

$$D_\mu E = -\frac{1}{8}\not{F}^{(4)}\Gamma_\mu E,$$

and we can build up an explicit solution $E = z^{-1/2}R_{S_+}R_{S_-}\xi$ as shown in the box that follows.

Explicit Form of the Killing Spinors

Firstly we restrict to Poincaré Killing spinors, i.e. those independent of the boundary coordinates of AdS. For AdS_2 this means that $\partial_t E = 0$. Thus the $\mu = t$ equation becomes

$$\frac{1}{2z}\Gamma^{01}E = \frac{1}{2}\Gamma^{0167}P_1(1 - P_2)\Gamma_t E,$$

which simplifies to the constraint

$$E = -\Gamma^{067}P_1P_2E.$$

The $\mu = z$ equation then reads $\partial_z E = -\frac{1}{2z}E$ which has the solution $E \propto z^{-1/2}$.

We then take a look at the first sphere. For $\mu \in S_+$ we get

$$D_\mu E = \frac{\sqrt{\alpha}}{2}\Gamma_\mu \Gamma^{2368} E, \tag{3.10}$$

where we have used the fact that $P_1 E = P_2 E = E$. The Killing spinors for S^2 are given in [?]. The equation

$$D_\mu \epsilon = \pm \frac{i}{2} e^a_\mu \Gamma_a \epsilon,$$

is solved by

$$\epsilon = \exp\left(\pm i \frac{\theta}{2}\Gamma^2\right) \exp\left(\frac{\phi}{2}\Gamma^{23}\right).$$

The θ equation is easy to solve, but the ϕ equation requires more work. One needs to Taylor expand the exponentials and use the fact that $(i\Gamma^2)^2 = -1$ and $\{i\Gamma^2, \Gamma^{23}\} = 0$. In our case, we replace $i\Gamma^2 \to \Gamma^{368}$ which still satisfies these conditions. Thus

$$R_{S_+} = \exp\left(\frac{\theta_+}{2}\Gamma^{368}\right) \exp\left(\frac{\phi_+}{2}\Gamma^{23}\right).$$

Then we take a look at the second sphere. For $\mu \in S_-$ we get

$$D_\mu E = \frac{\sqrt{1-\alpha}}{2}\Gamma_\mu \Gamma^{4578} E,$$

where $[\Gamma_\mu \Gamma^{4578}, R_{S_+}] = [D_\mu, R_{S_+}] = 0$, meaning that this is the equation which must be solved by R_{S_-} alone.

Finally, we arrive at the complete solution

$$E(\xi) = \frac{1}{\sqrt{z}} \exp\left(\frac{\theta_+}{2}\Gamma^{368}\right) \exp\left(\frac{\phi_+}{2}\Gamma^{23}\right) \exp\left(\frac{\theta_-}{2}\Gamma^{578}\right) \exp\left(\frac{\phi_-}{2}\Gamma^{45}\right)\xi, \tag{3.11}$$

with constant ξ obeying

$$-\Gamma^{067}\xi = P_1\xi = P_2\xi = \xi. \tag{3.12}$$

There are exactly 4 Killing spinors ξ_a, and we may take them to be orthonormal and normalised: $\xi_a \cdot \xi_b = \delta_{ab}$.

We need complex combinations $\mathcal{E}_i$ of the Killing spinors E_i above in order to perform fermionic T-duality. In general we may write these as

$$\mathcal{E}_i = b_{ia} E(\xi_a), \quad a = 1, ...4.$$

Then the commutativity condition (3.2) becomes

$$b_{ia} V^{\mu}_{ab} b_{jb} = 0, \quad V^{\mu}_{ab} \equiv \bar{E}_a \Gamma^{\mu} E_b. \tag{3.13}$$

For the AdS_2 case we can show that $V^{\mu} = 0$, using $\Gamma^{067} E = -E$, except for $\mu = t$, where

$$V^t_{ab} = E^T_a \Gamma^0_z \Gamma^0 E_b = -\xi^T_a \xi_b = \delta_{ab}.$$

So that the commutativity condition on the coefficients b_{ia} is simply $b_{ia} b_{ja} = 0$.

3.3.1 The case $\alpha = 1$: $AdS_2 \times S^2 \times T^6$

This is the simplest AdS_2 case. Here we can proceed in a similar fashion to [23]'s treatment of type IIB $AdS_3 \times S^3 \times T^4$. However, this case is even simpler as we only have two complexified Killing spinors.

With $P_2 E = E$, the $\mu \in S_+$ equation (3.10) simplifies to

$$D_{\mu} E = \frac{1}{2} \Gamma_{\mu} \Gamma^1 E,$$

and thus the resulting solution is given by

$$E(\xi) = \frac{1}{\sqrt{z}} \exp\left(\frac{\theta}{2} \Gamma^{21}\right) \exp\left(\frac{\phi}{2} \Gamma^{23}\right) \xi \tag{3.14}$$

with $\xi_a, \;\; a = 1, ..., 4$ still solving (3.12). We may write these as

$$(\xi_a)^\beta = \frac{1}{2}\delta_{2a-\beta-1} + \frac{1}{2}\delta_{2a-\beta+8} + \frac{1}{2}\delta_{2a-\beta+16} - \frac{1}{2}\delta_{2a-\beta+23}.$$

Where we note that each ξ has 4 non-zero entries at the positions given by the Kronecker-delta's, with $\beta = 1, ..., 32$ temporarily. Our complex combinations $\mathcal{E}_i = b_{ia}E(\xi_a)$ are specified by choosing[5]

$$b_1 = (1, 0, i, 0), \quad b_2 = (0, 1, 0, i),$$

which corresponds to

$$\mathcal{E}_1 = E_1 + iE_3, \quad \mathcal{E}_2 = E_2 + iE_4.$$

Then, after integrating (3.3), we arrive at

$$C = \frac{1}{z}\begin{bmatrix} i\cos\theta_+ - \sin\theta_+\sin\phi_+ & -\cos\phi_+\sin\theta_+ \\ -\cos\phi_+\sin\theta_+ & i\cos\theta_+ + \sin\theta_+\sin\phi_+ \end{bmatrix}.$$

To find the shift in the dilaton due to fermionic T-duality, we take the determinant of C, then (3.4) gives

$$\Delta\Phi = \Phi' - \Phi = -\log(z). \tag{3.15}$$

Note that the background we arrive at after doing a fermionic T-duality is complex. It also shifts the dilaton. To get back to the original background, which has $\Phi = 0$, we need to cancel this shift and somehow obtain a real background. Bosonic T-duality along the time direction (which is also the only boundary direction of AdS_2) does precisely this. It undoes the shift by giving us $\Phi'' - \Phi' = \log(z)$ and makes our background real. It also changes the metric in the following way:

$$ds^2_{AdS} = \frac{-dt^2 + dz^2}{z^2} \rightarrow -z^2dt^2 + \frac{dz^2}{z^2} = \frac{-dt^2 + dz'^2}{z'^2}, \tag{3.16}$$

where we have defined $z' = 1/z$ in the last step in order to bring the metric back to its original form.

[5]Note that choosing these complex combinations is not a well defined process and usually involves some trial and error.

Finally we consider the RR forms. The change in the bispinor F (3.5) has some real terms (which cancel the original flux (3.9)) and imaginary terms giving rise to a 2-form and a 4-form flux.

$$\begin{aligned} e^{\Phi'} \not{F}^{(2)'} &= i\Gamma^{14}, \\ e^{\Phi'} \not{F}^{(4)'} &= -i\Gamma^{0235} - i\Gamma^{1569} + i\Gamma^{1578}. \end{aligned} \tag{3.17}$$

Then bosonic T-duality along the x^0, x^4, x^6 and x^7 directions recovers the original RR flux. The total number of bosonic T-dualities is four, half of which must be along torus directions. This was also the case for $AdS_5 \times S^5$ in [22] and for $AdS_3 \times S^3 \times T^4$ in [23].

In earlier treatments of $AdS_2 \times S^2$ found in [124,125], only the coset σ-model $PSU(1,1|2)/U(1)^2$ is studied, rather than the critical string theory. Here the need for bosonic T-duality along torus directions was not noticed. However, this approach was extended to include non-coset fermions in [36], which forced the inclusion of T-duality along torus directions.

3.3.2 $AdS_2 \times S^2 \times T^6$ with other RR fluxes

Earlier we studied the case $\alpha = 1$ with only a $F^{(4)}$ turned on. There exist other combinations of fluxes with the same underlying geometry. One such case, considered in [36, 39, 126], has flux given by

$$\not{F}^{(2)} = -\Gamma^{01}, \quad \not{F}^{(4)} = \Gamma^{23}(\Gamma^{45} + \Gamma^{67} + \Gamma^{89}). \tag{3.18}$$

Showing self-duality for this background is no more difficult than the $F^{(4)}$ only case. We may write

$$S = -4\Gamma^{01}\Gamma_{11}(1 - P),$$

in terms of the projection operator

$$P = \frac{1}{4}(1 - \Gamma^{6789} - \Gamma^{4589} - \Gamma^{4567}).$$

Here, the Killing spinors are still given by (3.14), however, the constraint on ξ is now given by

$$\Gamma^0\Gamma_{11}P\xi = \xi.$$

We choose solutions so that

$$(\xi_a)^\beta = 0 \text{ for } \beta < a, \quad (\xi_a)^\beta > 0 \text{ for } \beta = a,$$

and then make the following complex combinations:

$$\mathcal{E}_1 = iE_1 + E_4, \quad \mathcal{E}_2 = iE_2 - E_3.$$

This gives

$$C = \frac{1}{z}\begin{bmatrix} ie^{i\phi_+}\sin\theta_+ & -i\cos\theta_+ \\ -\cos\theta_+ & -ie^{-i\phi_+}\sin\theta_+ \end{bmatrix},$$

which yields a shift in the dilaton of $\Delta\Phi = -\log(z)$. This is identical to (3.15), and is similarly cancelled by a bosonic T-duality along the time direction. The change in RR forms is

$$\begin{aligned} \Delta F^{(2)} =& \gamma^{01}, \\ \Delta F^{(4)} =& -\gamma^{23}(\gamma^{45} + \gamma^{67} + \gamma^{89}) - i\gamma^{1468} + i\gamma^{1479} + i\gamma^{1569} + i\gamma^{1578}. \end{aligned}$$

We can clearly see that the real terms cancel the original flux, as expected. Finally, acting with bosonic T-duality along the directions $t = x^0, x^4, x^6$ and x^8 returns us to (3.18).

3.3.3 The case $\alpha < 1$: $AdS_2 \times S^2 \times S^2 \times T^4$

Our generic complex combinations are given by $\mathcal{E}_i = b_{ia}E_a$. One can show that there does not exist a choice of b_{ia} which leads to a shift in the dilaton of the form $\Delta\Phi = \log(z) + \text{const}$. Hence, in this case, fermionic T-duality produces a shift in the dilaton which necessarily depends on the sphere coordinates $\theta_\pm$, $\phi_\pm$. Then bosonic T-duality along time (and some flat torus directions) cannot undo this shift.

To circumvent this problem, we allow bosonic T-duality to act along a complex Killing vector. To this end we introduce some new coordinates for one of the spheres. These coordinates arise from

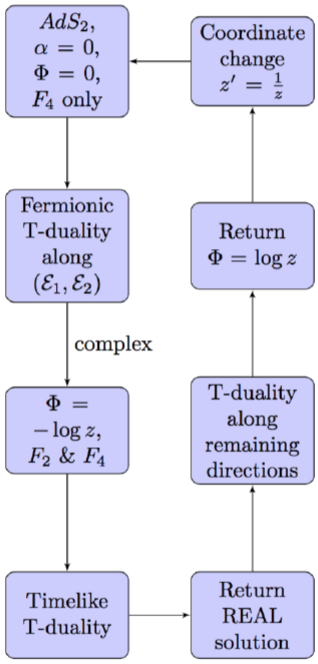

Figure 3.1: A summary of the T-dualisation process for $AdS_2 \times S^2 \times T^6$.

the form of the coset element used in [36][6]. Since S^2 is a symmetric space it may be described by a coset space G/H where the Lie algebra $\mathfrak{g}$ of G admits a $\mathbb{Z}_2$-grading (or automorphism). Then, as a vector space, $\mathfrak{g}$ can be decomposed into a direct sum of graded subspaces[7]

$$\mathfrak{g} = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(2)}.$$

In particular we have $S^2 = SU(2)/U(1)$ and

$$\mathfrak{su}(2) = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(2)},$$

where

$$\mathfrak{g}_{(0)} = \langle L_+ + L_- \rangle = \langle L_1 \rangle, \quad \mathfrak{g}_{(2)} = \langle L_+ - L_-, L_3 \rangle = \langle L_2, L_3 \rangle.$$

This coset space is parametrized by the coset element

$$g = e^{\lambda_+ L_+} e^{-\lambda_3 L_3} \in \mathfrak{su}(2),$$

[6]The algebra has been rescaled to avoid factors of $c = \sqrt{\alpha}$. Further discussion of related parametrisations occur in [125].

[7]This will be discussed in more detail in the next chapter.

where $L_\pm = iL_1 \pm L_2$, $L_n = \frac{1}{2i}\sigma_n$ and the σ_n are the Pauli matrices. Now we introduce the Maurer-Cartan 1-form which takes values in the Lie algebra $\mathfrak{g}$:

$$J = g^{-1}dg \in \mathfrak{su}(2),$$

with d denoting the exterior derivative. Using the $\mathbb{Z}_2$-automorphism we are able to decompose J, as a vector space, into $J = J_{(0)} + J_{(2)}$. Then $J_{(2)}$ is the projection of the Maurer-Cartan current onto $\mathfrak{g}_{(2)}$:

$$J_{(2)} = [g^{-1}dg]_{(2)} = e^{i\lambda_3}d\lambda_+ L_2 + d\lambda_3 L_3.$$

From this the metric is constructed as follows

$$ds^2_{S^+} = -2\mathrm{Tr}(J_{(2)}J_{(2)}) = d\lambda_3^2 + e^{2i\lambda_3}d\lambda_+^2.$$

The reason for constructing the metric using the trace of the Maurer-Cartan form is because it is the only 1-form that, by construction, satisfies the Maurer-Cartan equation. This ensures that our metric possesses and preserves the symmetries associated with the chosen Lie group G.

Comparing this metric to the metric in (3.8) allows us to show that the new coordinates $x^{\underline{2}} = \lambda_3$, $x^{\underline{3}} = \lambda_+$ are related to θ_+, ϕ_+ as follows

$$\begin{aligned} e^{i\lambda_3} &\equiv \cos\theta_+ + i\sin\theta_+\cos\phi_+, \\ \lambda_+ &\equiv e^{-i\lambda_3}\sin\theta_+\sin\phi_+ = \frac{\sin\phi_+}{\cot\theta_+ + i\cos\phi_+}. \end{aligned} \tag{3.19}$$

In what follows we write $e^3 = e^3_\mu dx^\mu = \sin\theta_+ d\phi_+$ and so on for the original coordinates. Then, underlined at indices will represent new coordinates: $x^{\underline{2}} = \lambda_3$, $x^{\underline{3}} = \lambda_+$. This change of coordinates preserves the volume form $e^2 \wedge e^3 = e^{\underline{2}} \wedge e^{\underline{3}}$. By changing coordinates we change the original real S^2 (with θ_+, ϕ_+) into a complex S^2. When T-duality is applied to this exceptional case we need to choose one of the new complex directions, together with the time-like direction and two torus directions, in order to illustrate self-duality. Therefore, we choose to T-dualise along λ_+ as our new complex direction.

The dual metric is given by

$$ds'^2_{AdS_2} = -z^2dt^2 + \frac{dz^2}{z^2}, \quad ds'^2_{S_+} = d\lambda_3^2 + e^{-2i\lambda_3}d\lambda_+^2, \tag{3.20}$$

and a shift in the dilaton

$$\Delta\Phi = \Phi'' - \Phi' = \log\left(ze^{-i\lambda_3}\right) = \log\left(\frac{z}{\cos\theta_+ + i\sin\theta_+ \sin\phi_+}\right).$$

Notice that the dual metric in terms of θ_+, ϕ_+ is no longer real. To recover a sphere we need to choose a new pair of real coordinates. Define θ', ϕ' by (3.19) with λ_3 replaced by $-\lambda_3$, i.e.

$$\begin{aligned} e^{i\lambda_3} &= \cos\theta' + i\sin\theta' \cos\phi', \\ \lambda_+ &= \frac{\sin\phi'}{\cot\theta' + i\cos\phi'}. \end{aligned}$$

This returns $ds'^2_{S_+} = d\theta'^2 + \sin^2\theta' d\phi'$. As before, in order to recover AdS_2 we need to invert the radial coordinate by defining $z' = 1/z$ which gives $ds'^2_{AdS_2} = (-dt^2 + dz'^2)/z'^2$.

For the fermionic T-duality we choose the following complex combinations

$$\mathcal{E}_1 = E_1 + iE_4, \quad \mathcal{E}_2 = -E_2 + iE_3.$$

The constant Killing spinors ξ_a which we input to get the E_a's are easy to find (using *Mathematica* for example), but are rather messy to display here. These constant spinors are ordered by the position of the first non-zero component where we have chosen the sign such that the first term is positive:

$$(\xi_a)^\beta = 0 \ \text{ for } \ \beta < a. \quad (\xi_a)^\beta > 0 \ \text{ for } \ \beta = a.$$

Then we use $\mathcal{E}_1$, $\mathcal{E}_2$ to calculate the matrix C

$$C = -2i\frac{\cos\theta_+ + i\sin\theta_+ \cos\phi_+}{z}\begin{bmatrix} -e^{-i\phi_-}\sin\theta_- & \cos\theta_- \\ cos\theta_- & e^{i\phi_-}sin\theta_- \end{bmatrix}$$

and find the change in the dilaton

$$\Delta\Phi = \Phi' - \Phi = \log\left(\frac{\cos\theta_+ + i\sin\theta_+ \cos\phi_+}{z}\right) + \log(-2i). \tag{3.21}$$

which cancels the bosonic shift found earlier perfectly (apart from the constant which may be absorbed).

Considering the change in the RR forms, we notice that a 2-form has appeared

$$e^{i\Phi'}F'^{(2)} = \sqrt{1-\alpha}\frac{(\sin\theta_+ - i\cos\theta_+\cos\phi_+)e^1\wedge e^2 + i\sin\phi_+ e^1\wedge e^3}{\cos\theta_+ + i\sin\theta_+\cos\phi_+} = -i\sqrt{1-\alpha}e^1\wedge e^{\underline{2}}.$$

Rewriting the 2-form in the new underlined at coordinates simplifies the expression greatly. The change in the 4-form $\Delta F^{(4)}$ has real terms which precisely cancel the original $F^{(4)}$, as required. It also has imaginary terms, greatly simplified by the new coordinates, and given by

$$e^{i\Phi'}F'^{(4)} = -i(e^1e^{\underline{3}}e^6e^8 + e^1e^{\underline{3}}e^7e^9) + i\sqrt{\alpha}(-e^0e^{\underline{2}}e^6e^7 + e^0e^{\underline{2}}e^8e^9) + i\sqrt{1-\alpha}(e^0e^{\underline{3}}e^4e^5).$$

Acting with bosonic T-duality on $F'^{(2)}$ and $F'^{(4)}$ in the usual way and along the directions $(t = x^0;\ \lambda = x^{\underline{3}}, x^7, x^8)$ gives

$$F''^{(2)} = 0,$$
$$F''^{(4)} = (e^0e^1e^6e^7 + e^0e^1e^8e^9) + \sqrt{\alpha}(e'^{\underline{2}}e'^{\underline{3}}e^6e^8 + e'^{\underline{2}}e'^{\underline{3}}e^7e^9) + \sqrt{1-\alpha}(-e^4e^5e^7e^8 + e^4e^5e^6e^9).$$

The new at coordinates $e'^{\underline{2}}$, etc. are interpreted as being related to the new metric ds'^2 (3.20) created by the bosonic T-duality. Thus $e'^{\underline{3}} = e'^{-i\lambda_3}d\lambda_+ \neq e^{\underline{3}}$. Notice that $e'^{\underline{2}} = -d\lambda_+ = -e^{\underline{2}}$ so that we preserve the volume form with this second coordinate change to the new coordinates $\theta' = x^{2'}$, $\phi' = x^{3'}$ we write this as $e'^{\underline{2}}\wedge e'^{\underline{3}} = e^{2'}\wedge e^{3'}$ Thereby recovering the original flux $F^{(4)}$ in (3.9).

3.4 $AdS_3 \times M$ Backgrounds

Here we will start with the $AdS_3 \times S^3 \times S^3 \times S^1$ background immediately. The simpler cases follow in much the same way as the simple cases for AdS_2, and there is nothing new to learn from them as a result. Also, the case $AdS_3 \times S^3 \times T^4$ was studied in [23] in the context of IIB string theory. Hence the focus on the $0 < \alpha < 1$ case here.

The metric is given by

$$ds^2_{AdS_3} = \frac{-dt^2 + dx^2 + dz^2}{z^2}$$

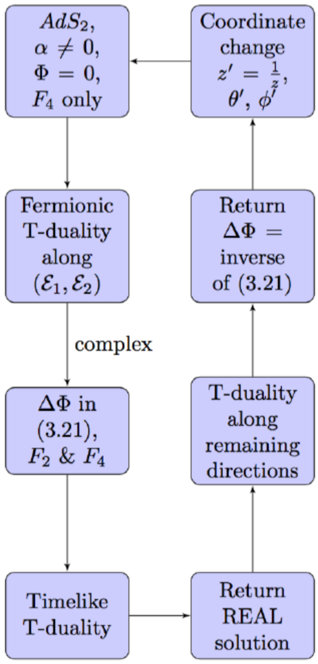

Figure 3.2: A summary of the T-dualisation process for $AdS_2 \times S^2 \times S^2 \times T^4$.

which has accompanying spin connection components $\omega_{t[02]} = \omega_{x[21]} = 1/z$. Following [?], we use nested coordinates for the S^3

$$ds^2_{S^3_\pm} = d\theta^2_\pm + \sin^2\theta_\pm(d\phi^2_\pm + \sin^2\phi_\pm d\psi^2_\pm)$$

with spin connection components $\omega_{\phi_+[34]} = -\cos\theta_+$, $\omega_{\psi_+[35]} = -\cos\theta_+\sin\phi_+$ and $\omega_{\psi_+[45]} = -\cos\phi_+$. The flux is given by

$$\begin{aligned} \not{F}^{(4)} =& 2(-\Gamma^{0129} + \sqrt{\alpha}\Gamma^{3459} + \sqrt{1-\alpha}\Gamma^{6789}) \\ =& -4\Gamma^{0129}(1-P) \end{aligned} \tag{3.22}$$

as in [121], where the projection operator is

$$P = \frac{1}{2}(1 + \sqrt{\alpha}\Gamma^{012345} + \sqrt{1-\alpha}\Gamma^{012678}).$$

Notice that for a bosonic T-duality along the x^9 direction, we return to fluxes considered in [127]. Furthermore, taking the limit $\alpha \to 1$ reduces this to the IIB $AdS_3 \times S^3 \times T^4$ case studied by O'Colgain [23].

This background has 16 supersymmetries, giving us 8 Poincaré Killing spinors and therefore 4 complex directions along which we may perform fermionic T-duality. We arrive at a solution $E = z^{-1/2} R_{S_+} R_{S_-} \xi$ in much the same way as we did for the AdS_2 case. For AdS_3 there is of course an extra boundary direction x, thus we have $\partial_t E = 0$ and $\partial_x E = 0$. The $\mu = t$ equation then gives us the constraint $E = \Gamma^{019} P E$, and the $\mu = x$ equation is identical. As before, the $\mu = z$ equation is solved by $E \propto z^{-1/2}$. Using $PE = E$, for $\mu \in S_+$ we have

$$D_\mu E = \frac{\sqrt{\alpha}}{2} \Gamma_\mu \Gamma^{01345} E$$

which is solved by a factor $R_{S_+} = e^{\frac{\theta_+}{2}\Gamma^{0145}} e^{\frac{\phi_+}{2}\Gamma^{34}} e^{\frac{\psi_+}{2}\Gamma^{45}}$. Similarly, for $\mu \in S_-$ we get

$$D_\mu E = \frac{\sqrt{1-\alpha}}{2} \Gamma_\mu \Gamma^{01678} E$$

which is solved by $R_{S_-} = e^{\frac{\theta_-}{2}\Gamma^{0178}} e^{\frac{\phi_-}{2}\Gamma^{67}} e^{\frac{\psi_-}{2}\Gamma^{78}}$ alone, since $[\Gamma_\mu \Gamma^{01678}, R_{S_+}] = [D_\mu, R_{S_+}] = 0$. The complete solution is

$$E(\xi) = \frac{1}{\sqrt{z}} \exp\left(\frac{\theta_+}{2}\Gamma^{0145}\right) \exp\left(\frac{\phi_+}{2}\Gamma^{34}\right) \exp\left(\frac{\psi_+}{2}\Gamma^{45}\right) \exp\left(\frac{\theta_-}{2}\Gamma^{0178}\right) \exp\left(\frac{\phi_-}{2}\Gamma^{67}\right) \exp\left(\frac{\psi_-}{2}\Gamma^{78}\right) \xi$$

with constant ξ obeying

$$\Gamma^{019} \xi = P\xi = \xi.$$

As mentioned earlier there are 8 of these spinors. We shall organise the orthogonal ξ_a so that

$$(\xi_a)^\beta = 0 \ \text{ for } \ \beta < a, \quad (\xi_a)^\beta = \frac{1}{2} \ \text{ for } \ \beta = a.$$

The commutativity relation (3.13) is only non-zero for $\mu = t, x$:

$$V^t = -1, \quad V^x = -\sqrt{\alpha}\sigma^2 \otimes \sigma^2 \otimes 1_{2\times 2} - \sqrt{1-\alpha} 1_{2\times 2} \otimes \sigma^1 \otimes 1_{2\times 2}.$$

After some trial and error, we arrive at the following complex combinations

$$\mathcal{E}_1 = E_1 - iE_8, \quad \mathcal{E}_2 = E_2 + iE_7, \quad \mathcal{E}_3 = E_3 - iE_6, \quad \mathcal{E}_4 = E_4 + iE_5.$$

This choice leads to

$$C = C_+(z, \theta_+, \phi_+, \psi_+)C_-(\theta_-, \phi_-, \psi_-)$$

where

$$C_+(z, \theta_+, \phi_+, \psi_+) = \frac{2i}{z}\sin\theta_+(\cos\phi_+ + i\sin\phi_+\cos\psi_+)$$

is a number, and

$$C_- = \begin{bmatrix} -c_\theta - is_\theta s_\phi(\hat{\alpha}c_\psi + \sqrt{\alpha}s_\psi) & -i\hat{\alpha}c_\phi s_\theta & -\sqrt{\alpha}c_\phi s_\theta & s_\theta s_\phi(\sqrt{\alpha}c_\psi - \hat{\alpha}s_\psi) \\ -i\hat{\alpha}c_\phi s_\theta & is_\theta s_\phi(\hat{\alpha}c_\psi - \sqrt{\alpha}s_\psi) - c_\theta & s_\theta s_\phi(\sqrt{\alpha}c_\psi + \hat{\alpha}s_\psi) & \sqrt{\alpha}c_\phi s_\theta \\ -\sqrt{\alpha}c_\phi s_\theta & s_\theta s_\phi(\sqrt{\alpha}c_\psi + \hat{\alpha}s_\psi) & c_\theta + is_\theta s_\phi(\hat{\alpha}c_\psi - \sqrt{\alpha}s_\psi) & i\hat{\alpha}c_\phi s_\theta \\ s_\theta s_\phi(\sqrt{\alpha}c_\psi - \hat{\alpha}s_\psi) & \sqrt{\alpha}c_\phi s_\theta & i\hat{\alpha}c_\phi s_\theta & c_\theta - is_\theta s_\phi(\hat{\alpha}c_\psi + \sqrt{\alpha}s_\psi) \end{bmatrix}$$

is a matrix with unit determinant. Here $s_\theta = \sin\theta_-$, $c_\theta = \cos\theta_-$, etc. Also, $\hat{\alpha} = \sqrt{1-\alpha}$. The change in the dilaton is then

$$\Delta\Phi = \Phi' - \Phi = 2\log\frac{2\sin\theta_+(\cos\phi_+ + i\sin\phi_+\cos\psi_+)}{z}. \tag{3.23}$$

As for the AdS_2 case with two spheres, we need to perform bosonic T-duality along some complex Killing vectors in order to undo the shift in the dilaton. From the parametrisation of the coset element in [36] we arrive at the metric

$$ds^2_{S^3_+} = d\lambda_3^2 + e^{2i\lambda_3}d\lambda_+^2 - e^{2i\lambda_3}d\lambda_-^2.$$

Notice that this is the generalisation of (3.20). Next, dualise along λ_+, λ_- and two AdS_3 directions t and x. Doing so gives us a shift in the dilaton of

$$\Delta\Phi = \Phi'' - \Phi' = 2\log(ze^{-i\lambda_3})$$

from which we read off

$$e^{i\lambda_3} = \sin\theta_+(\cos\phi_+ + i\sin\phi_+\cos\psi_+). \tag{3.24}$$

This duality changes the the metric $ds^2_{S^3_+}$ to $ds'^2_{S^3_+} = d\lambda_3^2 + e^{-2i\lambda_3}d\lambda_+^2 - e^{2i\lambda_3}d\lambda_-^2$. The dual metric is no longer real when expressed in the old coordinates θ_+, etc. So, just as before, in order to recover a sphere we must choose a different real slice given by θ', ϕ' and ψ' which are defined by (3.24) with λ_3 replaced by $-\lambda_3$. That is, $e^{-i\lambda_3} = \sin\theta'(\cos\phi' + i\sin\phi'\cos\psi')$.

Finally, we consider the changes made to the RR fields. The fermionic T-duality produces a 2-form flux which, as for the AdS_2 case, simplifies when written in the new complex coordinates. Again, using underlined flat indices for the new coordinates, we write $e^{\underline{3}} = d\lambda_3$. Then

$$\begin{aligned}\Delta F^{(2)} =& 2\sqrt{1-\alpha}\left[\cot\theta_+ e^2 e^3 + \frac{i(\cos\phi_+\cos\psi_+ + i\sin\phi_+)e^2e^4 - i\sin\psi_+ e^2 e^5}{\sin\theta_+(\cos\phi_+ + i\sin\phi_+\cos\psi_+)}\right] \\ =& 2i\sqrt{1-\alpha}e^2 e^{\underline{3}}. \end{aligned} \tag{3.25}$$

For the 4-form flux

$$\begin{aligned}\Delta F^{(4)} =& -2\sqrt{\alpha}e^0e^1e^9\left[\cot\theta_+ e^3 + \frac{i(\cos\phi_+\cos\psi_+ + i\sin\phi_+)e^4 - i\sin\psi_+ e^5}{\sin\theta_+(\cos\phi_+ + i\sin\phi_+\cos\psi_+)}\right] \\ &+ 2\sqrt{\alpha}e^3e^4e^5e^9 + 2\sqrt{1-\alpha}e^6e^7e^8e^9 - 2e^0e^1e^2e^9 \\ &+ 2e^2e^9\left[\cot\theta_+ e^4e^5 + \frac{-i(\cos\phi_+\cos\psi_+ + i\sin\phi_+)e^3e^5 - i\sin\psi_+ e^3e^4}{\sin\theta_+(\cos\phi_+ + i\sin\phi_+\cos\psi_+)}\right].\end{aligned}$$

Notice that the terms on the second line exactly cancel the original $F^{(4)}$. The first line contains $e^{\underline{3}}$, given by the square bracket, as in (3.25). The square bracket on the third line is just $ie^{\underline{4}}e^{\underline{5}}$, this is seen by noting that the volume form for the underlined coordinates should be the same as for the original coordinates, i.e.

$$[ie^{\underline{3}}] \wedge [ie^{\underline{4}} \wedge e^{\underline{5}}] = (\cot^2\theta_+ + ...)e^3 \wedge e^4 \wedge e^5 = -e^3 \wedge e^4 \wedge e^5.$$

Then, after the fermionic T-duality, the RR fluxes are

$$e^{\Phi'}F^{(2)'} = 2i\sqrt{1-\alpha}e^2e^{\underline{3}} \tag{3.26}$$
$$e^{\Phi'}F^{(4)'} = 2i\sqrt{\alpha}e^0e^1e^{\underline{3}}e^9 + 2\sqrt{1-\alpha}e^6e^7e^8e^9. \tag{3.27}$$

To get back to the original flux we bosonic T-dualise along t, x and $\lambda_\pm$ (i.e. directions $0, 1, \underline{4}, \underline{5}$) which leads to

$$e^{\Phi''}F^{(4)''} = -2e^0e^1e^2e^9 + 2\sqrt{\alpha}e'^{\underline{3}}e'^{\underline{4}}e'^{\underline{5}}e^9 + 2\sqrt{1-\alpha}e^6e^7e^8e^9.$$

Finally, we use that $e'^{\underline{3}}e'^{\underline{4}}e'^{\underline{5}} = e^{3'}e^{4'}e^{5'}$ to write $F^{(4)''}$ in terms of the final set of real coordinates θ', ϕ' and ψ'. Thus, recovering the original RR flux, (3.22).

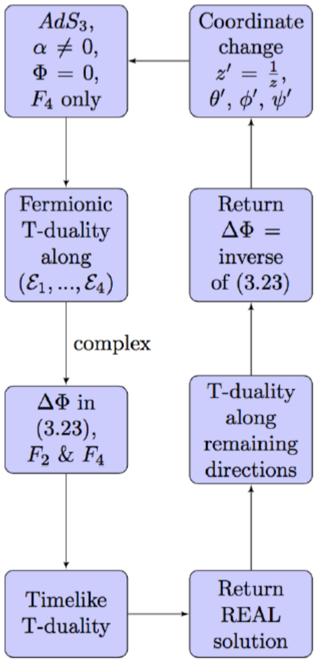

Figure 3.3: A summary of the T-dualisation process for $AdS_3 \times S^3 \times S^3 \times T^1$.

3.5 Combined Bosonic and Fermionic T-duality: In General

This section deals with proving the invariance of $AdS_n \times S^n \times M^{10-2n}$ backgrounds (and their exceptional extensions) under combined bosonic and fermionic T-duality without using the σ-model approach. In other words, this section generalises the treatment of these backgrounds, dealt with earlier on in this chapter, by applying T-duality transformations directly to the supergravity fields. We consider these geometries whilst being supported by various RR fluxes. This discussion extends the earlier results in [16, 22, 23, 101, 107] to the whole class of $AdS_n \times S^n \times M^{10-2n}$ super-backgrounds supported by RR fluxes.

3.5.1 Rules for Fermionic T-duality

Killing Spinors

The component supergravity fields along the bosonic directions are T-dualised using the original Buscher rules [12, 13, 129]. Generalising these rules to fermionic directions was given in [22]. Fermionic T-duality acts on the dilaton $\Phi(X)$ and the RR fields (which are p-forms) whilst leaving the metric and NSNS 2-form B flux invariant (see Chapter 2 for more details). In the fermionic case we dualise along directions specified by Grassmann-even[8] Killing spinors, $\Xi_\mu(X)$ (where μ labels their number), that generate Abelian superisometries. This means that the Killing spinors need to satisfy an additional condition called the commutativity condition

[8] Grassmann algebras are examples of supercommutative algebras. These algebras may be decomposed into even and odd variables which satisfy a graded version of commutativity. In particular, even elements commute whilst odd elements anticommute.

$$\Xi_\mu \Gamma_A \Xi_\nu = 0 \quad \text{for all} \;\; A, \mu, \nu \;\; \text{with} \;\; A = 0, 1, ..., 9. \tag{3.28}$$

This condition has important consequences for complex Killing spinors; the only type of spinor for which non-trivial solutions may be obtained. As a result, they are associated with complex Grassmann-odd directions in superspace. The Killing spinor equations have the following form

$$\begin{aligned}\left(\partial_M - \frac{1}{4}\Omega_M^{AB}(X)\Gamma_{AB} + \frac{1}{8}\not{F}\mathcal{E}_M^A(X)\Gamma_A\right)\Xi = 0,\\ \frac{1}{16}\Gamma_A \not{F}\Gamma^A \Xi = 0,\end{aligned} \tag{3.29}$$

where $\Omega_M^{AB}(X)$ and $\mathcal{E}_M^A(X)$ are the spin connection and the bosonic vielbeins of the 10-dimensional background. Thus, the covariant derivative is given by $\nabla_M := \partial_M - \frac{1}{4}\Omega_M^{AB}\Gamma_{AB}$. The contribution to the RR fluxes is given by $\not{F}$:

$$\not{F} = \begin{cases} e^\Phi \left(\frac{1}{2}F_{AB}^{(2)}\Gamma^{AB}\Gamma_{11} + \frac{1}{4!}F_{ABCD}^{(4)}\Gamma^{ABCD}\right) & \text{type IIA} \\ -\frac{e^\Phi}{2}(1+\Gamma^{11})\left(iF_A^{(1)}\Gamma^A\sigma^2 + \frac{1}{3!}F_{ABC}^{(3)}\Gamma^{ABC}\sigma^1 + \frac{i}{2\cdot 5!}F_{ABCDE}^{(5)}\Gamma^{ABCDE}\sigma^2\right) & \text{type IIB} \end{cases}$$

The Killing spinor equations in (3.29) are obtained by requiring that the supersymmetry variations of the gravitino and dilatino vanish. They are also determined by the geometry of the background and the chosen RR fluxes (see Chapter 2 for a detailed discussion). The second equation in (3.29) is used together with the integrability requirement for the first equation in (3.29) to obtain a projector $\mathcal{P}_{8(n-1)}$, singling out the $8(n-1)$ fermionic isometries of the backgrounds of interest. Once this projector has been found, the second equation in (3.29) is identically satisfied.

Fermionic T-duality Rules

As noted in Chapter 2, the main object of interest when performing a fermionic T-duality is the scalar field C which is used to calculate the changes in the fields. We arrive at this matrix C once we have solved the Killing spinor equations which yield symmetry preserving Killing spinors. Then, after making complex combinations of these spinors such that they satisfy (3.28), we are able to construct C from

$$\partial_M C_{\mu\nu} = \begin{cases} E_M^A \bar{\Xi}_\mu \Gamma_A \Gamma^{11} \Xi_\nu & \text{type IIA} \\ E_M^A \bar{\Xi}_\mu \Gamma_A \sigma^3 \Xi_\nu & \text{type IIB} \end{cases}$$

The matrix $C = C_{\mu\nu}(X)$ is formed by the components of the NSNS 2-form B field along the Abelian fermionic isometries (i.e. $\Theta = 0$) [22]

$$d\theta^\mu \wedge d\theta^\nu B_{\mu\nu}(X, \Theta)|_{\Theta=0} := d\theta^\mu \wedge d\theta^\nu C_{\mu\nu}(X), \tag{3.30}$$

where C depends on the bosonic directions X only. Once we have obtained C then we can find the dilaton as follows

$$\Delta\Phi = \Phi' - \Phi = \frac{1}{2}\log(\det C), \tag{3.31}$$

and the change in the RR fluxes

$$\Delta F = \not{F}' - \not{F} = 8\Xi_\mu\mu(C^{-1})^{\mu\nu}\Xi_\mu\nu\Gamma, \tag{3.32}$$

where Γ is a certain product of Γ-matrices which we use to split the fermionic $E^{(1,2)}$ currents into four pieces. This corresponds to splitting the superalgebra generator $\mathcal{Q}$ into Q, $\hat{Q}$, S and $\hat{S}$ respectively.

Background	RR Flux	Γ
$AdS_5 \times S^5$	F_5 flux	1
$AdS_3 \times S^3 \times T^4$	F_3 flux	$-\Gamma^{23}$
$AdS_3 \times S^3 \times S^3 \times S^1$	F_3 flux	$-\Gamma^{23}$
$AdS_3 \times S^3 \times S^3 \times S^1$	F_4 flux	Γ^{239}
$AdS_2 \times S^2 \times T^6$	F_3 flux	$-\Gamma^{23}$
$AdS_2 \times S^2 \times T^6$	F_2 & F_4 fluxes	$\Gamma^{11}\Gamma^{123}$
$AdS_2 \times S^2 \times T^6$	F_4 flux	Γ^1
$AdS_2 \times S^2 \times S^2 \times T^4$	F_4 flux	Γ^{239}

Table 3.2: Table displaying the Γ corresponding to each background and their various fluxes.

Explicit Form of the Killing Spinors

Berkovits and Maldacena describe a direct way to find the explicit form of the Killing spinors associated with the anti-commuting fermionic isometries along which we dualise the supergravity fields [22]. This method suggests extracting the form of the Killing spinors directly from the corresponding components of the fermionic currents J_Q associated with the generators Q of the superisometry algebra. By definition, the Killing spinors must satisfy the commutativity condition (3.28) and the Killing spinor equations (3.29). Concretely, the Killing spinors are given by the components of the matrix $J_\alpha^\beta(|y|, y^{\hat{a}}, \lambda_3)$ in

$$J_{Q_\alpha}|_{\Theta=0} = d\theta^\mu J_\mu^\alpha(|y|, y, \lambda_3) = d\theta^\mu e^{-B} Q_\mu e^B|_{Q_{\alpha,\hat{\theta}=\hat{\xi}}=0} = d\theta^\mu \Xi_\mu^\alpha, \tag{3.33}$$

where

$$e^B := e^{\hat{\theta}^\alpha \hat{Q}_\alpha + \hat{\xi}^\alpha \hat{S}_\alpha}|y|^D e^{-\lambda_3 L_3} e^{-\rho_\alpha^\beta R_\beta^\alpha}. \tag{3.34}$$

Note that $\Theta = 0$ implies that we consider components proportional to the generators of our isometries. The commutativity condition is obtained by differentiating (3.33) (using the exterior derivative) to get

$$0 = d\Xi_\mu + [e^{-B}de^B, \Xi_\mu \mu]|_{\Theta=0},$$
$$e^{-B}de^B|_{\Theta=0} = \Omega^{\hat{a}\hat{b}}(y/|y|)R_{\hat{a}\hat{b}} + J_D(|y|)D + J_{L_3}(\lambda_3)L_3.$$

Note that the index μ labels a given Killing spinor. From the structure of e^B (3.34) and the commutation relations

$$[D,Q] = \frac{1}{2}Q,$$
$$[R_{\hat{a}},Q] = -\frac{s^2}{2}Q\Gamma_{\hat{a}}\Gamma^4\mathbb{P},$$
$$[L_3,Q] = \frac{i}{2}Q,$$

we have the following explicit form of the Killing spinors in question[9]

$$\Xi_\mu^\alpha = J_\mu^\alpha(|y|,y,\lambda_3) = |y|^{-\frac{1}{2}}e^{\frac{i}{2}c\lambda_3}\mathcal{O}_\mu^\alpha(y^{\hat{a}}/|y|), \tag{3.35}$$

where $\mathcal{O}_\mu^\alpha(y^{\hat{a}}/|y|) := (e^{s\mathbb{P}\Gamma_{\hat{a}}\Gamma_4 y^{\hat{a}}/(2|y|)})_\mu^\alpha$ is a *Spin* $(n+1)$ matrix associated with the coset $s^n \approxeq SO(n+1)/SO(n)$ and $\mathbb{P} := \mathbb{P}_+\mathcal{P}_{8(n-1)}$ is the projection matrix that singles out the $2(n-1)$ anti-commuting isometries $Q = Q\mathbb{P}$ for each case of $AdS_n \times S^n \times M^{10-2n}$. The projector $\mathbb{P}_\pm$ is given by $P_\pm := \frac{1}{2}(1 \pm i\Gamma^{0123})$. By definition,

$$\mathcal{O}^T\Gamma_4\mathcal{O} = \Gamma_4\mathbb{P}. \tag{3.36}$$

The structure of the scalar field $C_{\mu\nu}$ can be read off from (3.30) to give

$$B_{\mu\nu}|_{\Theta=0} = iJ_\mu^\gamma\Gamma^4_{\gamma\delta}J_\nu^\delta(|y|,y,\lambda_3) = C_{\mu\nu}.$$

The explicit form of the matrix C can be found using (3.35) and (3.36). Then,

[9]The coset representative for $AdS_2 \times S^2 \times T^6$ is obtained from the representative for $AdS_2 \times S^2 \times S^2 \times T^4$ by taking the limit $c \to 0$. To do this, rescale the coordinates $\lambda \to c\lambda$, $\lambda_3 \to c\lambda_3$, and $\rho_\beta^\alpha \to s\rho_\beta^\alpha$ finally taking the limit $c \to 0$. Performing this limit makes the metric of the second sphere become flat. This decouples the sphere from the $AdS_2 \times S^2$ subspace. As a result, it re-compactifies into a T^2 which then forms the required T^6 for the $AdS_2 \times S^2 \times T^6$ background.

$$\begin{aligned} C_{\mu\nu} &= iJ^\gamma_\mu \Gamma^4_{\gamma\delta} J^\delta_\nu, \\ &= i\left(|y|^{-\frac{1}{2}} e^{\frac{i}{2}c\lambda_3} \mathcal{O}^\gamma_\mu\right) \Gamma^4_{\gamma\delta} \left(|y|^{-\frac{1}{2}} e^{\frac{i}{2}c\lambda_3} \mathcal{O}^\delta_\nu\right), \\ &= i|y|^{-1} e^{ic\lambda_3} (\mathcal{O}\Gamma^4\mathcal{O})_{\mu\nu}, \\ &= i|y|^{-1} e^{ic\lambda_3} (\Gamma^4\mathbb{P})_{\mu\nu}, \end{aligned} \tag{3.37}$$

and the inverse is given by

$$(C^{-1})^{\mu\nu} = -i|y|^{-1} e^{ic\lambda_3} (\mathbb{P}\Gamma^4)^{\mu\nu}. \tag{3.38}$$

From (3.37), the dilaton shift is

$$\Delta\Phi = \frac{1}{2}\log(\det C) = -(n-1)\log|y| + i(n-1)c\lambda_3. \tag{3.39}$$

Notice that this dilaton shift has the exact form of all the cases studied so far. The change in the RR fields is found using (3.38)

$$\Delta F = 8J_\mu (C^{-1})^{\mu\nu} J_\nu = -8i\mathbb{P}\Gamma^4\Gamma = -(1 + i\Gamma^{0123})\not{F}, \tag{3.40}$$

in terms of Γ. Thus, (3.40) is the change in RR fields after a fermionic T-duality transformation for any of the cases dealt with thus far.

Explicit Form of the RR Fluxes

Here we consider the change in the RR fluxes (3.40) supporting the exceptional backgrounds, $AdS_n \times S^n \times S^n \times M^{10-3n}$. For $n = 3$, we have $AdS_3 \times S^3 \times S^3 \times S^1$ and we may consider the type IIB theory supported by an F_3 flux[10]

$$F_3 = \frac{1}{3}(\varepsilon_{cba} e^a \wedge e^b \wedge e^c + \frac{R_{AdS}}{R_+} \varepsilon_{\hat{c}\hat{b}\hat{a}} e^{\hat{a}} \wedge e^{\hat{b}} \wedge e^{\hat{c}} + \frac{R_{AdS}}{R_-} \varepsilon_{c'b'a'} e^{a'} \wedge e^{b'} \wedge e^{c'}),$$

In this case $\Gamma = -\Gamma^{23}$ as in the non-exceptional ($\alpha = 0$) case. This supergravity background arises due to the intersection of $D1$-branes and $D5$-branes [130]. The solution preserves 16 supersymmetries. If instead we chose to dualise along the ϕ^9-direction (or the S^1-coordinate), then we would arrive at a type IIA background supported by an F_4 flux [121]

[10]The radii for each subspace have the following values: $R_{AdS} = 1$, $R_+ = 1/\sqrt{\alpha}$, and $R_- = 1/\sqrt{1-\alpha}$.

$$F_4 = d\phi^9 \wedge \frac{1}{3}(\varepsilon_{cba} e^a \wedge e^b \wedge e^c + \frac{R_{AdS}}{R_+}\varepsilon_{\hat{c}\hat{b}\hat{a}} e^{\hat{a}} \wedge e^{\hat{b}} \wedge e^{\hat{c}} + \frac{R_{AdS}}{R_-}\varepsilon_{c'b'a'} e^{a'} \wedge e^{b'} \wedge e^{c'}),$$

where $\Gamma = \Gamma^{239}$, and we use the projector $\mathcal{P}_{16}$. This supergravity background arises due to the dimensional reduction of the eleven-dimensional $AdS_3 \times S^3 \times S^3 \times T^2$ solution and it represents the near-horizon geometry of two $M5$-branes and an $M2$-brane which intersect over a line [131].

For $n = 2$, the exceptional case has the same F_4 flux as the $\alpha = 0$ case. Here, we have a rank-8 projector which may be decomposed into a product of two rank-16 projectors as in [121], i.e. $\mathcal{P}_8 = P_1 P_2$. This solution is obtained by dimensional reduction of the eleven-dimensional $AdS_2 \times S^2 \times S^2 \times T^5$ solution and it represents the near-horizon geometry for the intersection of two $M2$-branes and four $M5$-branes [132]. Re-numbering the F_4 flux components in accordance with [121] such that $0, ..., 3$ represents the directions along which we dualise (with $2, 3, 8$ and 9 the T^4 directions). Parametrize one of the spheres by $x^7 = \lambda_3$ and $x^1 = \lambda_+$ and the second sphere by x^5 and x^6, then we can write

$$\not{F}_4 = 4 P_1 P_2 \Gamma^{0492},$$

with

$$P_1 = \frac{1}{2}(1 + \Gamma^{9238}), \quad P_2 = \frac{1}{2}(1 + \sqrt{\alpha}\Gamma^{047123}\sqrt{1-\alpha}\Gamma^{045698}),$$

where $\Gamma = \Gamma^{239}$. In all the cases considered here, the shifts (3.39) and (3.40) are undone, precisely, by the corresponding bosonic T-duality.

3.5.2 Compensating Bosonic T-duality

In Chapter 1, we discussed how the complete Buscher rules for the bosonic case form part of a larger $O(D, D)$ symmetry group. It is this group that forms the basis for generalized geometry [133]. In the cases that we have considered, the antisymmetric NSNS B field vanishes and the metric is diagonal, this greatly simplifies the rules. Let $\mathcal{I}$ be the set of directions along which we dualise, the new metric has the following components

$$G'_{tt} = \frac{1}{G_{tt}}, \quad t \in \mathcal{I},$$

and remains unchanged in all other directions. The dilaton shift is given by the determinant of this block of the matrix

$$\Delta\Phi = -\frac{1}{2}\log(\det G_{MN}) = -\frac{1}{2}\sum_{t\in\mathcal{I}} \log G_{tt}.$$

Finally, we can write down the change in the RR fields as

$$\not{F}'' = \left(\prod_{t \in \mathcal{I}} c_t \Gamma^t \right) \not{F}',$$

where t refers to at directions here, $\not{F}'$ is the result of the fermionic T-duality, and

$$c_t := \begin{cases} -i & t = 0 \\ 1 & t \neq 0. \end{cases}$$

Note that for a non-trivial dilaton extra factors need to be included, this was dealt with by Fukuma et al. [81]. Time-like T-duality results in imaginary forms [83], however, the overall sign appearing after performing combined T-duality is not physical. It only depends on the order in which we apply our dualities.

Concerning the backgrounds under consideration, the bosonic T-dualisation was performed along the directions labelled by $t = \{0, ..., 3\}$. The T-dualised RR fields are then

$$\not{F}'' = -i\Gamma^{0123} \not{F}'.$$

On substituting this back into $\not{F}' = \not{F} + \Delta F$ with the shift ΔF coming from the fermionic T-duality, we find that the combined bosonic-fermionic T-duality leaves the RR fields intact, i.e. $\not{F}'' = \not{F}'$. Thus these backgrounds are T-self-dual.

3.6 Summary

This chapter considered type II supergravity backgrounds $AdS_2 \times S^2 \times T^6$ and $AdS_d \times S^d_+ \times S^d_- \times T^{10-3d}$ for $d = 2, 3$, including a general treatment of all the less supersymmetric backgrounds simultaneously. These backgrounds are interesting because they are not maximally supersymmetric, unlike $AdS_5 \times S^5$, which has been widely studied. Furthermore, their self-duality has not been thoroughly studied in the supergravity context. These backgrounds have, however, been studied (in various ways) using coset σ-models. We have shown that these non-maximally supersymmetric backgrounds are self-dual under a series of bosonic and fermionic T-dualities. It is important to stress that it is not possible to avoid performing bosonic T-duality along some of the torus directions, and for the exceptional cases (i.e. $S_+ \times S_-$) we have to bosonic T-dualise along some complex Killing vectors along one of the spheres. Furthermore, these self-dual backgrounds are integrable. This is a useful fact as it provides evidence favouring the proposed link between integrability and self-duality.

Part III

Green-Schwarz σ-model

Introduction to Part III

Part III contains the body of original research focusing on the σ-model perspective. The goal is to investigate whether backgrounds possessing less than maximal supersymmetry are self-dual under a sequence of bosonic and fermionic T-dualities. Two ideas are present in the part. In particular, we consider the AdS backgrounds of the form $AdS_d \times S^d \times T^{10-2d}$, $(d = 2, 3)$ and $AdS_d \times S^d \times S^d \times T^{10-3d}$, $(d = 2, 3)$. Then we prove that $AdS_5 \times S^5$ is self-dual without fixing κ-symmetry gauge. To understand where κ-symmetry plays a role, it is helpful to recall the bosonic string worldsheet. There, we quantize the fields X^μ using oscillators that satisfy the following commutation relations[11]

$$[a^\mu_k, a^{\dagger\nu}_p] = \delta_{kp}\eta^{\mu\nu}.$$

Notice that the zero modes (i.e. $\mu = \nu = 0$) are negative norm states. These states are undesirable and can be removed through use of the gauge symmetry. Conformal symmetry is part of the gauge symmetry which we use to remove negative norm states. Moving onto the superstring, we find that negative norm fermionic states arise as well. To remove these negative norm states we need a larger gauge symmetry (as there are more degrees of freedom). This gauge symmetry is called κ-symmetry and it removes half of the fermionic degrees of freedom[12]. This new symmetry plays an important role, as we will see, when studying the self-duality of backgrounds that may be written as a graded coset structure (i.e. the supersymmetric cousin of a bosonic symmetric space [36]). The study of $AdS_5 \times S^5$ was greatly facilitated by the observation that the Green-Schwarz action for this background could be written as a $\mathbb{Z}_4$-graded coset[13] [126, 138]

$$\frac{PSU(2, 2|4)}{SO(1, 4) \times SO(5)}.$$

The motivation for studying string theory directly, using worldsheet methods, arose shortly after the AdS/CFT correspondence between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$, $D = 4$ super Yang-Mills theory [6, 139–142] was described. This provided an opportunity to prove that $AdS_5 \times S^5$ was an exact string solution, to write down the corresponding 2-dimensional conformal field theory and

[11] Here we assume a mostly plus metric.

[12] The existence of κ-symmetry first appeared in superparticle actions in [134–137]. Kappa-symmetry is a gauge symmetry, larger than the conformal group, which is used to remove negative norm bosonic states. Then, κ-symmetry removes negative norm fermionic states. However, the term κ-symmetry was not used in the original references, being introduced later by Townsend [108].

[13] The special projective unitary group $PSU(N)$ is the isometry group of a complex projective space, just as the projective orthogonal group is the isometry group for real projective spaces.

to find the full string spectrum [138]. This idea (of using coset structures) has been used heavily in integrability [143] and attempts to understand fermionic T-duality, for example: Beisert, Ricci, Tseytlin and Wolf [21] work entirely with the coset action, gauge fixing a particular κ-symmetry at the start. They found that the dual σ-model amounts to a different choice of κ-symmetry gauge [36].

The two formulations both have their strengths and weaknesses. The main advantage of using the RNS formalism is that quantization becomes straightforward since the action is free[14] in a flat background [118, 118]. There are three important disadvantages. Firstly, the RNS formalism cannot describe superstrings on backgrounds with RR fields present because the RR vertex operators become complicated. Secondly, the symmetry of the spectrum is not manifest. Finally, the extension of the RNS formalism to curved spacetimes is not obvious because there is a lack of spacetime covariance [108]. The GS formalism does allow a covariant extension to curved backgrounds through the existence of κ-symmetry. Furthermore, the GS superstring action may be defined classically in any background satisfying the supergravity equations of motion [118]. For background fields satisfying these equations of motion, the GS action is classically invariant under κ-symmetry, which is required for the removal of non-physical fermionic degrees of freedom. The main disadvantage of the GS formalism is that quantization becomes nontrivial [145, 146]. It is important to stress that in the GS formalism κ-symmetry invariance requires that the background fields are on-shell[15], whereas for the RNS formalism it is quantum Weyl invariance that ensures this self-consistency condition [108]. This is illustrated in Figure 3.4.

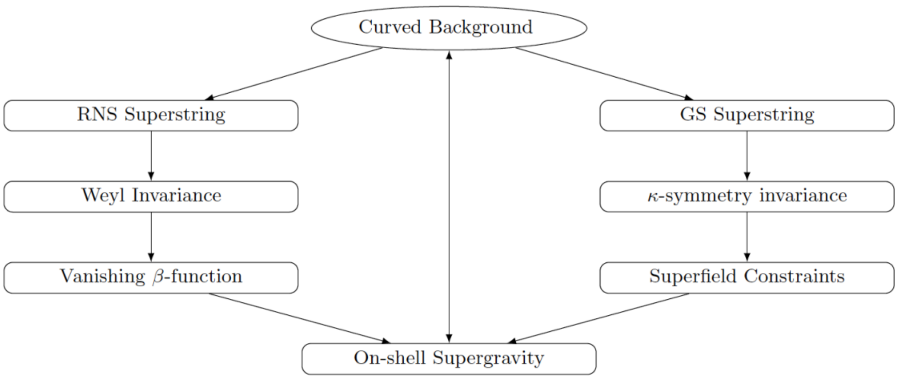

Figure 3.4: Different superstring formulations require curved backgrounds to be on-shell [108].

Consider, for the moment, the maximally supersymmetric background $AdS_5 \times S^5$ which is supported by a (Hodge) self-dual RR 5-form flux. As a result, following the discussion above, the RNS formalism is not useful in a straightforward way[16]. The GS formalism, being manifestly supersymmetric, appears to be sufficient for non-vanishing RR fields [115]. Although the formal expression for the GS superstring action (in superspace) for a generic type IIB background was presented in [116], it is not very practical for finding the explicit form of the superstring action in terms of the coordinate fields (X, Θ)[17]. The maximal supersymmetry of the $AdS_5 \times S^5$ background suggests that an alternative approach is needed, one which

[14] By free we mean that the action does not possess any interaction terms.

[15] By on-shell we mean that the supergravity equations of motion must be obeyed. Since supersymmetry relates bosons to fermions, it means that you can turn any bosonic intial condition into a fermionic intial condition, implying equality of the number of degrees of freedom for the bosonic and fermionic fields.

[16] Although the non-local RR vertex operator is known in flat space [114], it is insufficient for describing the complete form of the RNS string action in curved backgrounds

[17] For a bosonic background, the corresponding $D = 10$ type IIB superfields need to be determined explicitly. This is a very complicated problem, not yet solved for any non-trivial cases.

exploits the special properties of this solution [117]. Over a decade ago, such an alternative approach to constructing superstring actions was developed, which combines the advantages from the RNS and GS formalisms [118] (see also [119, 120]). Following the RNS approach, our action reduces to a free action for a flat background in which quantization becomes straightforward. This hybrid approach uses spacetime spinor variables as fundamental fields, as in the GS formalism, allowing for simple RR vertex operators [118]. The hybrid formulation is not without disadvantages either, namely, 10-dimensional Lorentz invariance is no longer manifest [118]. For an alternative approach, see also the *Pure Spinor Formalism* by Berkovits [171].

Chapter 4 uses a general approach, where we verify the self-duality of Green-Schwarz supercoset σ-model $AdS_5 \times S^5$ without gauge fixing κ-symmetry. However, we start with a very general introduction to the most important concepts used throughout Chapter 4 and 5. Furthermore, in Chapter 5, we also consider superstrings on the exceptional backgrounds $AdS_d \times S^d \times S^d$ for $(d = 2, 3)$.

Chapter 4

Supercoset Models

4.1 Introduction

This chapter introduces the techniques used to show T-self-duality using coset σ-models. We begin with a very gentle introduction to the coset geometry of the S^3 manifold. This allows one to introduce graded Lie algebras, coset representatives and the currents that will build our Lagrangians. Simple examples are provided to help expose concepts and permit intuition. Then, we provide a more formal introduction to supercoset models and the general setup we will be using, including the T-duality procedure. Finally we consider the example of $AdS_5 \times S^5$, an exciting example with important consequences for self-duality.

4.2 The Coset Geometry of S^3

As an example, we consider the coset geometry of the 3-sphere, S^3, as a warm up which will lay bare the techniques used throughout this chapter. Since the rest of the chapter is technically dense and very messy to write down, this section will also serve an introduction to the methods used and will provide some intuition. The backgrounds considered in Chapter 3 are called symmetric or at the very least semi-symmetric spaces. Spaces of this nature may be described by a coset

$$\frac{G}{H} := \{gH|g \in G\},$$

where G has the interpretation of being the isometry group and H is the isotropy group[1]. The key idea is that any Lie group may be described by a manifold. The elements of the corresponding Lie algebra (i.e. the generators of infinitesimal transformations) are the tangent space elements which generate transformations along the flat directions (i.e. in the tangent plane)[2]. In the end, the curved coordinates

[1]The isometry group captures all symmetries for which the given metric space will be invariant. The isotropy group forms a subgroup of G that fixes a given point on H. The group is given by $H = \{g \in G|gh = h\}$.

[2]Roughly speaking, the group H is "divided" out, meaning that the symmetries of the group H do not affect the overall geometry of the S^3 and therefore the metric.

on the manifold are related to the flat coordinates on the tangent space by the *vielbein*:

$$dy^a = e_\mu{}^a dx^\mu,$$

where the vielbein components are $e_\mu{}^a$. The Latin indices represent tangent space directions whilst the Greek indices represent curved directions. The 3-sphere, S^3, is a symmetric space and may be described by the following coset[3]:

$$S^3 = \frac{SO(4)}{SO(3)}.$$

This means that $g \sim gh$ (i.e. they are equivalent) for all $h \in SO(3)$ and $g \in SO(4)/SO(3)$. Points on S^3 correspond to equivalence classes of the coset $SO(4)/SO(3)$, and not to the elements g. However, we may also represent each point on S^3 by an $SO(4)$ vector. Transformations in the $SO(4)$ group rotate points on the S^3 to other points on the S^3. One might ask why the 3-sphere is not fully described by the $SO(4)$ group. The reason is that, at each point on the curved manifold S^3, there exists a tangent space. The tangent space at point p on S^3 consists of all rotations about p, which are described by $SO(4)$ vectors. Therefore, there is an entire set of rotations, in the tangent plane at any p, which leaves the vector at that point unchanged with respect to the global S^3 geometry. This is why we need to remove the degrees of freedom which are equivalent to rotations in the tangent space, to remove redundancy, and accurately capture the geometry of the 3-sphere. Cosets are the objects which allow us to do precisely this: they allow us to remove unwanted degrees of freedom. Furthermore, this is the reason that points on the 3-sphere are described by equivalence classes. Note that the 3-sphere is invariant under the action of the $SO(4)$ group [147].

The 3-sphere may be parametrized by a 4-vector, $(\sigma, \vec{v})$. The scalar fields, $(\sigma, \vec{v})$, transform as an $SO(4)$ 4-vector and are subject to the constraint [147]

$$\sigma^2 + \vec{v} \cdot \vec{v} = R^2.$$

In many cases, however, it is more useful to define a set of boosts (which are $SO(4)$ transformations), $L_{\vec{v}}$, which send a given reference point $(\sigma, \vec{v}) = (R, \vec{0})$ into a general point. To illustrate this action explicitly, we choose a representation of the $SO(4)$ group, writing the transformations as 4×4 orthogonal matrices: $O_{\alpha\beta}(p)$ for all $p \in SO(4)$. This is given by

$$\begin{aligned} \sigma &= O_{44}(L_{\vec{v}})R \\ v_i &= O_{i4}(L_{\vec{v}})R, \end{aligned}$$

where are indices run over $(4, 1, 2, 3)$ with $i = 1, 2, 3$ only. As an example, consider spherical polar coordinates. Then the point $(\sigma, \vec{v}) = (R, \vec{0})$ gets transformed into

[3]In general, any n-sphere can be written as $S^n = SO(n+1)/SO(n)$.

$$\begin{bmatrix} R\cos\theta \\ R\sin\theta\cos\phi \\ R\sin\theta\sin\phi\cos\psi \\ R\sin\theta\sin\phi\sin\psi \end{bmatrix} = L_{\vec{v}} \begin{bmatrix} R \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

when we choose $L_{\vec{v}}$ to be

$$L_{\vec{v}} = e^{-i\psi L_{23}} e^{-i\phi L_{12}} e^{-i\theta L_{41}}. \tag{4.1}$$

It must be noted that there is an arbitrariness in the choice of the operators $L_{\vec{v}}$[4]. In this example we used spherical polar coordinates, but some other transformation may be used, the important feature is that any choice of operator must satisfy the algebra

$$[L_{\alpha\beta}, L_{\gamma\delta}] = \delta_{\beta\gamma}L_{\alpha\rho} - \delta_{\alpha\gamma}L_{\beta\rho} + \delta_{\alpha\rho}L_{\beta\gamma} - \delta_{\beta\rho}L_{\alpha\gamma},$$

where the operator $L_{\alpha\beta} = -L_{\beta\alpha}$ is the generator of infinitesimal rotations in the $\alpha\beta$-plane. Notice that the coset representative (4.1)contains the generator L_41. This might seem odd at first since, for the 2-sphere, the infinitesimal generators were in the planes perpendicular to the chosen direction. Here we have fixed the direction 4 and so we might expect to have planes orthogonal to the 4-direction. But two things are happening. Firstly, in 4-dimensions orthogonality does not look the same as it does in 3-dimensions (i.e. our intuition may get us into trouble). Secondly, we might have chosen the following transformation

$$\begin{bmatrix} R \\ R\sin\theta\cos\phi \\ R\sin\theta\sin\phi\cos\psi \\ R\sin\theta\sin\phi\sin\psi \end{bmatrix} = L_{\vec{v}} \begin{bmatrix} R \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Notice that the 4-direction remained invariant. For this case we would certainly have a very different $L_{\vec{v}}$ as compared to (4.1). However, given that our choice transforms our 4-vector to a general 4-vector where the 4-direction depends on one of the $SO(3)$ coordinates θ, it is therefore, not unusual that it appears in (4.1) alongside θ. There is another instance where arbitrariness arises. In four dimensions we have the following planes

$$\begin{array}{ccc} 41 & 42 & 43 \\ \cancel{14} & 12 & 13 \\ \cancel{24} & \cancel{21} & 23 \\ \cancel{34} & \cancel{31} & \cancel{32}, \end{array}$$

[4]This point is often glossed over in the main body of the literature. Another way make this point clear is to notice that, choosing a coset representative is not unique. There is always some arbitrariness.

where the crossed out planes are repetitions (i.e. $14 \leftrightarrow 41$). From the third row only plane-23 remains, so it appears in (4.1). The second row has two options, the 12-plane or 13-plane. Only one of these need to be chosen, either choice will work and we have chosen the 12-plane. The first row presents the most options. Here we have the $4i$-planes ($i = 1, 2, 3$) to choose from. Again, either choice will satisfy the algebra. We have chosen the 41-plane.

Explicitly, an $SO(4)$ transformation may be written as

$$e^{-i\theta L_{41}} = \begin{bmatrix} \cos\theta & 0 & 0 & \sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta & 0 & 0 & \cos\theta \end{bmatrix}.$$

Applying an $SO(4)$ transformation, T, to the 3-sphere will carry the point with coordinates (θ, ϕ, ψ) into the point (θ', ϕ', ψ'). In general, the coordinate transformation is given by

$$O_{\alpha 4}(L'_{\vec{v}}) = O_{\alpha\beta}(T)O_{\beta 4}(L_{\vec{a}}) = O_{\alpha 4},(TL_{\vec{v}})$$

where $O_{\alpha\beta}$ is a 4×4 orthogonal matrix and $O_{\alpha 4}$ and $O_{\beta 4}$ are column vectors. For an orthogonal transformation in the 123-subspace (i.e. $S \in SO(3)$), we have

$$O_{\alpha 4}(S) = \delta_{alpha4}.$$

As a result, we may write

$$O_{\alpha 4}(L'_{\vec{v}}S) = O_{\alpha 4}(gL'_{\vec{v}}),$$

which implies that $gL_{\vec{v}} = L'_{\vec{v}}S$ (because $gL_{\vec{v}}$ and $L'_{\vec{v}}S$ have the same representation). Since we can find $L'_{\vec{v}}$ once we know what (θ', ϕ', ψ') are, the $SO(3)$ transformation

$$S = L'^{-1}_{\vec{v}}TL_{\vec{v}},$$

becomes well defined [147]. The coset formulation gives a well defined geometric description from which geometrical objects, like the metric and vielbeins, may be found using group theory.

To compute the basis vectors for the 3-sphere consider the differential [147]

$$J = L^{-1}_{\vec{v}}dL_{\vec{v}}. \tag{4.2}$$

where $J \in \mathfrak{so}(3)$ and d is the exterior derivative. This might seem mysterious at first glance, therefore let us first consider the following example.

Example: Translations

Suppose we want to do a translation along the x-direction from x to $x+a$. Then the function $f(x)$ becomes $f(x+a)$. The transformation may be written as

$$T(a)f(x) = e^{a\frac{\partial}{\partial x}f(x)} = f(x+a),$$

where $e^{a\frac{\partial}{\partial x}}$ is an element of the group of translations and $\frac{\partial}{\partial x}$ is the generator of infinitesimal translations, which lives in the Lie algebra of the group. The transformation is parametrized by a. In general, we have the following

$$de^{f(x)} = e^{f(x)}df,$$

where, in the case of a translation this becomes

$$d\left(e^{a\frac{\partial}{\partial x}}\right) = e^{a\frac{\partial}{\partial x}}da\left(\frac{\partial}{\partial x}\right).$$

The Lie algebra element is $\frac{\partial}{\partial x}$, the infinitesimal translation is da and the group element is $e^{a\frac{\partial}{\partial x}}$. Then we can write

$$e^{-a\frac{\partial}{\partial x}}de^{a\frac{\partial}{\partial x}} = da\left(\frac{\partial}{\partial x}\right).$$

Notice that this last equation has the same form as the differential in (4.2). It tells us that the left hand side produces an infinitesimal translation da along the x-direction (right hand side).

This differential has a special name, it is called the Maurer-Cartan 1-form, and is special because it satisfies the Maurer-Cartan equation. One-forms satisfying the Maurer-Cartan equations may be used to construct a metric that preserves the isometries of the Lie group. Computing (4.2) we find

$$\begin{aligned}
J =& L_{\vec{v}}^{-1}dL_{\vec{v}} \\
=& e^{i\theta L_{41}}e^{i\phi L_{12}}e^{i\psi L_{23}}de^{-i\psi L_{23}}e^{-i\phi L_{12}}e^{-i\theta L_{41}} \\
=& -d\theta L_{41} - d\phi(L_{12}\cos\theta + L_{42}\sin\theta) \\
& - d\psi(L_{23}\cos\phi + (L_{13}\cos\theta - L_{43}\sin\theta)\sin\phi) \\
=& -d\theta L_{41} - d\phi\sin\theta L_{42} + d\psi\sin\theta\sin\phi L_{43} \\
& - d\phi L_{12} - d\psi\cos\phi L_{23} + d\phi\cos\theta\sin\phi L_{31}
\end{aligned} \tag{4.3}$$

From the expression for J we can extract the following coefficients

$$e_\mu^a = \begin{array}{c} \\ \theta \\ \phi \\ \psi \end{array} \begin{array}{c} \begin{array}{ccc} 41 & 42 & 43 \end{array} \\ \begin{bmatrix} -1 & 1 & 1 \\ 0 & -\sin\theta & 0 \\ 0 & 0 & \sin\theta\sin\phi \end{bmatrix} \end{array} \tag{4.4}$$

where the labelling of rows and columns is indicated [147]. The elements in (4.4) are precisely the components of the vielbein ${e_\mu}^a dx^\mu$. Note that small displacements are generated by the generators which are Lie algebra valued. These generators are labelled by the plane (ij) that they rotate in. Thus, (4.4) tells us which generator generates a given small displacement. As an example, making a small rotation in the 41-plane means you will change the value of θ. From the vielbeins, we can write down the metric as follows

$$\begin{aligned} ds^2 =& (d\theta e^\alpha_\theta + d\phi e^\alpha_\phi + d\psi e^\alpha_\psi)^2 \\ =& d\theta^2 + \sin\theta^2 d\phi^2 + \sin\theta^2 \sin\phi^2 d\psi^2 \end{aligned} \tag{4.5}$$

which is expected for S^3 in polar coordinates.

4.3 The General Setup

In this section we recall some basic facts about superstring coset σ-models. We also lay the foundation required to study the T-dualisation of these σ-models. The Green-Schwarz action of a superstring that propagates in a 10-dimensional background is given by [116]

$$S = -\frac{T}{2}\int_\Sigma (\star\mathcal{E}^A \wedge \mathcal{E}^B \eta_{AB} + 2\kappa B_2), \tag{4.6}$$

where T denotes the string tension, Σ represents the 2-dimensional worldsheet with curved metric $h_{pq}(\tau,\sigma)$ having Lorentz signature such that the corresponding worldsheet Hodge duality operation $\star$ squares to one ($\star^2 = 1$) when acting on 1-forms[5]. The $\mathcal{E}^A = \mathcal{E}^A(X,\Theta)$ are vector space supervielbeins for all $A, B = 0, ..., 9$ where (X,Θ) are target space coordinates (10 Grassmann-even (or bosonic) coordinates X and 32 Grassmann-odd (or fermionic) coordinates Θ). The metric η_{AB} is the 10-dimensional target space Minkowski metric. Additionally, there are spinor supervielbeins $\mathcal{E}^{\hat\alpha} = \mathcal{E}^{\hat\alpha}(X,\Theta)$ for $\hat\alpha, \hat\beta = 1, ..., 32$ which also describe the geometry of the full 10-dimensional superstring. Finally, B_2 is the NSNS 2-form flux and we consider models for which B_2 has vanishing field strength at $\Theta = 0$, or $dB_2|_{\Theta=0} = 0$.

We want to study backgrounds which are symmetric or semi-symmetric[6] . These manifolds $\mathcal{M}$ may be described by a coset

[5]In local coordinates (i.e. (τ,σ)) on σ, $\star\mathcal{E}^A \wedge \mathcal{E}^B = \sqrt{-\det h_{rs}} h^{pq} \mathcal{E}^A_p \wedge \mathcal{E}^B_q$

[6]Semi-symmetric backgrounds have vanishing NSNS flux and a Lie algebra which admits a $\mathbb{Z}_4$-grading with accompanying automorphism $\Omega : G \to G$ and fixed point set H. This means that $\Omega^4 = 1$ and $\Omega(H) = H$.

$\mathbb{Z}_4$-graded Coset Superspaces				
Background Type	**d**	**Superspace**	**Coset**	$n_b + n_f$
$\alpha = 0$	5	$AdS_5 \times S^5$	$\frac{PSU(2,2\|4)}{SO(1,4)\times SO(5)}$	$10 + 32$
	3	$AdS_3 \times S^3$	$\frac{PSU(1,1\|2)\times PSU(1,1\|2)}{SU(1,1)\times SU(2)}$	$6 + 16$
	2	$AdS_2 \times S^2$	$\frac{PSU(1,1\|2)}{SO(1,1)\times U(1)}$	$4 + 8$
$0 < \alpha < 1$	3	$AdS_3 \times S^3 \times S^3$	$\frac{D(2,1;\alpha)\times D(2,1;\alpha)}{S0(1,2)\times SO(3)\times SO(3)}$	$9 + 16$
	2	$AdS_2 \times S^2 \times S^2$	$\frac{D(2,1;\alpha)}{S0(1,1)\times SO(2)\times SO(2)}$	$6 + 8$

Table 4.2: The cosets corresponding to the various backgrounds for $0 \leq \alpha < 1$. Here n_b and n_f are the number of bosonic and fermionic coordinates, respectively.

$$\mathcal{M} := \frac{G}{H},$$

as mentioned in the previous section. The Green-Schwarz action describes a semi-symmetric space, therefore, it admits a $\mathbb{Z}_4$-automorphism and there exists a truncation of the Green-Schwarz action to a supercoset σ-model[7].

4.3.1 Maurer-Cartan Forms and $\mathbb{Z}_4$-graded Coset Superspaces

The $\mathbb{Z}_4$-automorphism (or grading) $\Omega : G \to G$ induces an automorphism on the corresponding Lie superalgebra $\mathfrak{g}$ of G, which we will also denote by $\Omega : \mathfrak{g} \to \mathfrak{g}$. This automorphism implies that we may decompose the Lie algebra $\mathfrak{g}$ as follows

$$\mathfrak{g} \approxeq \mathfrak{g}_{(0)} + \mathfrak{g}_{(1)} + \mathfrak{g}_{(2)} + \mathfrak{g}_{(3)}.$$

As a result, the eigenspaces of may also be decomposed as follows

$$\Omega(V_m) = i^m V_{(m)} \text{ for } V_{(m)} \in \mathfrak{g}_{(m)}.$$

The Lie algebra $\mathfrak{g}$ contains the generators which generate all possible transformations that leave the metric invariant. The superalgebra elements satisfy the commutators: $[\mathfrak{g}_{(m)}, \mathfrak{g}_{(n)}] \subseteq \mathfrak{g}_{(m+n \mod 4)}$, where $\mathfrak{g}_{(0)}$ is the Lie algebra of H. Table 4.2 contains all the $\mathbb{Z}_4$-graded coset Superspaces considered herein. Note that the $d = 5$ coset describes the full superstring. For $d = 2, 3$, the cosets listed in Table 4.2 describe those subsectors of the full superstring theory in which the non-supersymmetric fermions have been removed.

There is also a $\mathbb{Z}_2$-grading on $\mathfrak{g}$ because we are dealing with a superspace and this means that we have both bosonic and fermionic generators. This divides all the generators into two subgroups, one containing the bosonic generators $\mathfrak{g}_{(0)}$ and $\mathfrak{g}_{(2)}$, and one containing the fermionic generators $\mathfrak{g}_{(1)}$ and $\mathfrak{g}_{(3)}$. For a detailed discussion on the general properties of the Lie superalgebra associated with the Lie supergroups

[7]For backgrounds which posses at least 16 supersymmetries, their corresponding σ-model can be viewed as a κ-symmetry gauge fixing of the full superstring.

considered in Table 4.2, see [148].

To define the Maurer-Cartan form we consider maps of the form $g : \Sigma \to G$ from the 2- dimensional worldsheet Riemann surface Σ onto G. Then, we introduce the pull-back[8] to Σ via g of the Maurer-Cartan form by

$$J := g^{-1}dg,$$

where the exterior derivative is given by d and in our conventions it acts from the right. By construction, J is a $\mathfrak{g}$-valued differential 1-form, which satisfies the Maurer-Cartan equation

$$dJ - J \wedge J = 0.$$

The Maurer-Cartan Equation

We constructed the general Maurer-Cartan form J to satisfy the Maurer-Cartan equation. In this aside we show how this can be proved. Start with the definition

$$J = g^{-1}dg$$

and take the derivative

$$d(g^{-1}dg) = dg^{-1} \wedge dg + g^{-1}d^2g = -g^{-1}dgg^{-1} \wedge dg = -g^{-1}dg \wedge g^{-1}dg$$

which is just $dJ = J \wedge J \implies dJ + JJ = 0$ and we used that $dg^1 = (g^{-1}dg)g^{-1}$ and $d^2g = 0$.

To gain some intuition about why J is a 1-form, consider the following. We may write

$$J = g^{-1}dg = f_u dx^\mu$$

where the f_μ are the components of the one form$J$. Because d is an exterior derivative, J is a 1-form and under a change in coordinates, f_μ transforms exactly like something with a lower index should. However, it deserves a special name because it is not a number. It is an element of the Lie algebra $\mathfrak{g}$, that is, a matrix. These forms are called Lie algebra valued (one) forms. Then pullback in the usual way, for example, if x^μ are spacetime coordinates and σ^A are the worldsheet coordinates, then

$$f_A = f_\mu \frac{\partial x^\mu}{\partial \sigma^A}.$$

[8] Roughly speaking, suppose you have a map defined by $\phi : X \to Y$ and f is a form defined on Y , then the pull-back $\phi * (f) : Y \to X$ is the form on X whose value at $x \in X$ is the value of f at $\phi(x) \in Y$. That is, the form on Y defines a form on X.

The 1-form J is invariant under *global left* G-transformations[9] $g_0 g \to g$ for $g_0 \in G$. The $\mathbb{Z}_4$- automorphism $\Omega : \mathfrak{g} \to \mathfrak{g}$ allows us to decompose J into the eigenspaces of Ω according to

$$J = J_{(0)} + J_{(1)} + J_{(2)} + J_{(3)}, \text{ with } \Omega(J_{(m)}) = i^m J_{(m)}.$$

The map Ω is the set of all $\mathbb{Z}_4$ transformations. Hence, the algebra can be thought of as a vector space and as a result the algebra may be decomposed into the eigenspaces of the map Ω as shown above. This is equivalent to a matrix acting on some vector space where you are able to use the eigenvectors of the matrix as a basis. Under local right H -transformations[10] $gh \to g$ for $h \in H$, the $J_{(0)}$ term is a $g_{(0)}$-valued *connection* 1-form (i.e. $J_{(0)} \in Lie(H)$). The H-transformations do not move physical points around in the spacetime (i.e. recall that these are transformations in the tangent space at each point which therefore leaves the point invariant with respect to H). Therefore, there is some redundancy in this description.

Using bosonic and fermionic supervielbeins$J_{(m)}$ where $m = 1, 2, 3$, the supercoset action may be defined. It has the following form

$$S = -T \int_{\Sigma} \mathcal{L}_{G/H} = -\frac{T}{2} \int_{\Sigma} \text{Str}(\star J_{(2)} \wedge J_{(2)} + J_{(1)} \wedge J_{(3)}) \tag{4.7}$$

where Str denotes the supertrace. It is compatible with the $\mathbb{Z}_4$-grading

$$\text{Str}(V_{(m)} V_{(n)}) = 0 \tag{4.8}$$

for $V_{(m)} \in \mathfrak{g}_{(m)}$ and $m + n \mod 4$. For the non-exceptional cases, the relative coefficients of the two terms in (4.7) are fixed by κ-symmetry. However, for the exceptional cases these coefficients are not fixed, therefore lacking κ-symmetry invariance [125, 149]. Instead, the coefficients are fixed by integrability conditions belonging to the respective σ-model. Comparing (4.7) to the Green-Schwarz action in (4.6) we can see that the Wess-Zumino term in (4.7), i.e. the second term in the action, is given by B_2 in (4.6). Thus

$$B_2 = \frac{1}{2} \text{Str}(J_{(1)} \wedge J_{(3)}).$$

The $\mathfrak{g}_{(0)}$ - valued forms are scalars (or invariant) under the Z$\mathbb{Z}_4$ - automorphism because the automorphism takes G into G and $\mathfrak{g}_{(0)} \in H$. Therefore, these are the Lie algebra elements that generate displacements in the coordinates, for example. These elements form the *connection* and transform as follows

$$J_{(0)} \to h^{-1} J_{(0)} h + h^{-1} dh.$$

The $J_{(m)}$'s for $m = 1, 2, 3$ transform adjointly as follows

[9]By global we mean that we use the same G transformation at each point on the manifold.
[10]By local we mean that a different transformation is applied at each point on the manifold.

$$J_{(m)} \to h^{-1} J_{(m)} h.$$

These Lie algebra elements do not generate coordinate displacements, they act more like internal symmetries. The physical fields will take values in the coset superspace $G/H = gH|g \in G$, thus the corresponding action (geometry) must be invariant under such local H - transformations. This is a reminder that the H- transformations do not move physical fields around. In turn the action may only contain $J_{(m)}$ for $m = 1, 2, 3$. These are precisely those terms that combine to form Casimirs[11] in order to return an action with the correct symmetries for the problem being studied. The coset removes all trivial rotations (i.e. those leaving a point invariant).

The coset G/H is parametrised by d_b bosonic local coordinates X and d_f fermionic local coordinates ϑ, where $d_b + d_f = \dim(G/H) = \dim(G) - \dim(H)$. The maps considered here are

$$(\mathbb{X}, \vartheta) : \Sigma \to G/H.$$

The bosonic supervielbein is given by $J_{J(2)}$ whilst the fermionic supervielbeins are given by $J_{(1)}$ and $J_{(3)}$.

4.3.2 Choosing a $\mathbb{Z}_4$-grading

The $\mathbb{Z}_4$-grading and its decomposition into Abelian sub-isometries must be chosen with care. If an inappropriate choice is made one cannot successfully apply the T-duality transformations. Thus, making the proof of self-duality complicated, if at all possible. To write down the grading first decompose the R-symmetry generators into $R = (R_{(0)}, R_{(2)})$ with $R_{(0)} \in \mathfrak{g}_{(0)}$ and $R_{(2)} \in \mathfrak{g}_{(2)}$. Then the $\mathbb{Z}_4$-grading we use is written as follows:

$$\begin{aligned} \mathfrak{g}_{(0)} &:= \langle P + K, M, R_{(0)} \rangle, \quad & \mathfrak{g}_{(2)} &:= \langle P - K, D, R_{(2)} \rangle \\ \mathfrak{g}_{(1)} &:= \langle Q - S, \hat{Q} - \hat{S} \rangle, \quad & \mathfrak{g}_{(3)} &:= \langle Q + S, \hat{Q} + \hat{S} \rangle. \end{aligned} \tag{4.9}$$

The specific decomposition of R will be discussed in the next section. The sub-algebras $\mathfrak{g}_{(0)}$ and $\mathfrak{g}_{(2)}$ contain bosonic symmetry generators, whilst $\mathfrak{g}_{(1)}$ and $\mathfrak{g}_{(3)}$ contain the fermionic symmetry generators.

4.3.3 Schematic Form of the Superconformal Algebra

The T-dualisation of the action (4.7) is performed along some bosonic and fermionic directions. These T-duality transformations correspond to an (anti-) commuting subgroup of isometries belonging to the underlying supercoset space. These isometries are identified through a choice of basis obtained from the Lie superalgebra $\mathfrak{g}$ of G. Associated with this basis is the superconformal group on the Minkowski

[11] Casimirs are combinations of Lie algebra elements that commute with everything else in the Lie algebra. For example, consider $SO(3)$. Here, the algebra is made up of $\{L_x, L_y, L_z\}$ and the Casimir is $\vec{L} \cdot \vec{L}$.

Symmetries and their Generators			
Symmetry	**Generator**	**N**	**Type**
Translations	P	$d-1$	$R^{1,d-2}$
Lorentz Boosts	M	$\frac{1}{2}(d-1)(d-2)$	$R^{1,d-2}$
Conformal	K	$d-1$	$R^{1,d-2}$
Dilatation	D	1	$R^{1,d-2}$
R-symmetry	R	2	$AdS_d \times S^d$
Fermionic	$Q, \hat{Q}, S, \hat{S}$	$2(d-1)$	$AdS_d \times S^d$

Table 4.4: Symmetries and their number of associated generators, N.

conformal boundary, $R^{(1,d2)}$ of the AdS_d space [36]. We may describe the basis for $\mathfrak{g}$ schematically as

$$\begin{gathered}[P,K] \sim D+M, \quad [D,P] \sim P, \quad [D,K] \sim K, \quad [M,P] \sim P \\ [M,K] \sim K, \quad [M,M] \sim M, \quad [R,R] \sim R, \end{gathered} \tag{4.10}$$

where we have chosen not to display vanishing commutators.

In the superconformal extension of (4.10) we find the fermionic generators listed in Table 4.4. These are the complex supersymmetry and superconformal generators [36]. The hatted and ordinary generators are related by Hermitian conjugation. The non vanishing commutators are given, schematically, by

$$\begin{aligned}&[D,Q] \sim Q\ , \quad [M,Q] \sim Q\ , \quad [K,Q] \sim \hat{S}\ , \quad [R,Q] \sim Q+\alpha\hat{Q}\ , \\ &[D,S] \sim S\ , \quad [M,S] \sim S\ , \quad [P,S] \sim \hat{Q}\ , \quad [R,S] \sim S+\alpha\hat{S}\ ,\end{aligned} \tag{4.11}$$

and similarly for the hatted generators. The anti-commutators are

$$\begin{gathered}\{Q,\hat{Q}\} \sim P\ , \quad \{S,\hat{S}\} \sim K\ , \quad \{Q,\hat{S}\} \sim \alpha R\ , \quad \{\hat{Q},S\} \sim \alpha R \\ \{Q,S\} \sim D+M+R\ , \quad \{\hat{Q},\hat{S}\} \sim D+M+R\ .\end{gathered} \tag{4.12}$$

Note that the α in the above (anti-) commutators is precisely the same one appearing in Table 4.2. The non-exceptional cases are obtained from the exceptional cases (i.e the last two entries in Table 4.2) by taking the limit $\alpha \to 0$. Therefore $\mathfrak{g} = \langle P, K, D, M, R, Q, \hat{Q}, S, \hat{S}\rangle$.

4.3.4 Coset Representative and Associated Current

For the non-exceptional cases ($\alpha = 0$), the form of the supersymmetry algebra (4.10), (4.11) and (**??**) imply that the generators P and the complex supercharges Q are in involution[12]. Thus, the maximal Abelian subalgebra of $\mathfrak{g}$ is just $\langle P, Q\rangle$, this means that the (anti-) commuting isometries of the G/H σ-

[12]If two things are in involution, they commute. This means that they can define an Abelian algebra.

model are associated with $\langle P, Q\rangle$. For the exceptional cases ($\alpha \neq 0$), the situation is trickier. The maximal Abelian subalgebra of $\mathfrak{g}$ is now generated by P and Q in addition to some R-symmetry generators, denoted by L_+. The latter generator is complex with Hermitian conjugate denoted by L. For $d = 2$ there is one such generator given by $L_+ \equiv L_+^1$, whilst for $d = 3$ there are two generators given by $L_+ \equiv L_+^{1,2}$. Thus, the (anti-) commuting isometries in the exceptional cases are associated with $\langle P, Q, L_+\rangle$. Then

$$[L_+^1, L_-^1] \sim L_3 \sim [L_+^2, L_-^2].$$

With respect to the $\mathbb{Z}_4$-grading (4.9), we find that $L_+^1 + L_-^1$ and $L_+^2 - L_-^2$ belong to $\mathfrak{g}_{(0)}$. On the other hand, $L_+^1 - L_-^1$, $L_+^2 + L_-^2$ and L_3 belong to $\mathfrak{g}_{(2)}$. This is the grading we chose here. Using this information, we are able to see that in order to perform T-dualisation on the supercoset action (4.7) along the isometries that we have already mentioned, we take the supercoset representation $\mathfrak{g}$ in the form following [21,22,125]

$$g \;:=\; \mathrm{e}^{xP+\theta Q+\sqrt{\alpha}\,\lambda_+ L_+}\mathrm{e}^{B}\mathrm{e}^{\xi S} \;, \quad \mathrm{e}^{B} \;:=\; \mathrm{e}^{\hat{\theta}\hat{Q}+\hat{\xi}\hat{S}}|y|^{D}\mathrm{e}^{-\sqrt{\alpha}\,\lambda_3 L_3}\Lambda_\alpha(y) \;, \tag{4.13}$$

where x are the coordinates of the Minkowski boundary and $|y|$ is related to the radial direction in AdS_d. In the non-exceptional case ($\alpha = 0$) the coordinates y parametrize the sphere S^d. In the exceptional cases ($\alpha \neq 0$), one S^d is parametrized by y and the second by λ_+ and λ_3. These coordinates, λ_+ and λ_3, are assumed to be complex. The specific form of the $\Lambda(y)$ will depend on the underlying geometry that is used. The Grassman-odd directions of the coset superspace are parametrized by $2(d-1)$ fermionic coordinates $(\theta, \hat{\theta}, \xi, \hat{\xi})$. The form of (4.13) was achieved using *local right H - transformations*. Also, because P, Q and L_+ are in involution our choice of coset representative guarantees that the action (4.7) depends on x, θ and L_+ alone. It depends on the latter coordinates through their derivatives dx, $d\theta$ and dL_+.

As we have mentioned the proof of self-duality of the supercoset σ-models (4.7) under bosonic and fermionic T-duality has up till now been performed by fixing κ-symmetry gauge [36,38,39,65,108,125,126]. The most convenient choice is setting $\xi = 0$ [21, 22, 125]. However, there is a problem, if the supercoset model has been gauge fixed with respect to the associated superstring action, then κ-symmetry has already been used to set the non-supersymmetric fermions to zero. As a result, κ-symmetry gauge fixing can no longer be used in the T-dualisation procedure. The situation is more severe when dealing with the exceptional backgrounds where the rank of the κ- symmetry is zero (i.e. none of the non-supersymmetric fermions can be set to zero) [37,125,149]. Thereby motivating our choice to study these σ-models without fixing κ-symmetry gauge. All fermionic coordinates in (4.9) will be considered. With respect to (4.13), the Mauer Cartan current has the following form

$$J \;=\; g^{-1}\mathrm{d}g \;=\; \mathrm{e}^{-\xi S}J^{(0)}\mathrm{e}^{\xi S} + \mathrm{d}\xi S \;, \tag{4.14}$$

where $J_{(0)}$ is the current at $\xi = 0$. For a detailed list of the components of J and their explicit forms, see [36]. However, schematically the components of J are

$$\begin{aligned}
J_P &= J_P^{(0)} \ , \quad J_Q = J_Q^{(0)} \ , \quad J_{L_+} = J_{L_+}^{(0)} \ , \\
J_D &= J_D^{(0)} + J_Q^{(0)}\xi \ , \quad J_M = J_M^{(0)} + J_Q^{(0)}\xi \ , \\
J_R &= J_R^{(0)} + J_Q^{(0)}\xi \ , \quad J_{L_3} = J_{L_3}^{(0)} + \alpha J_Q^{(0)}\xi \ , \\
J_{\hat{Q}} &= J_{\hat{Q}}^{(0)} + J_P^{(0)}\xi \ , \quad J_{\hat{S}} = J_{\hat{S}}^{(0)} + \alpha J_{L_+}^{(0)}\xi \ , \\
J_K &= J_{\hat{S}}^{(0)}\xi + \alpha J_{L_+}^{(0)}\xi^2 \ , \quad J_{L_-} = \alpha J_{\hat{Q}}^{(0)}\xi + \alpha J_P^{(0)}\xi^2 \ , \\
J_S &= \mathrm{d}\xi + (J_D^{(0)} + J_M^{(0)} + J_R^{(0)} + \alpha J_{L_3}^{(0)})\xi + J_Q^{(0)}\xi^2 \ .
\end{aligned} \tag{4.15}$$

Here the current J depends at most on ξ quadratically. This is very important as it simplifies the T-dualisation procedure significantly.

4.3.5 The T-duality Procedure

Proceeding as in [22,150], the non-exceptional cases are T-dualised along x and θ. The exceptional cases are T-dualised along x, and $_+$, following [125]. In the standard procedure [12,13,129] we start with the supercoset σ-model[13] and then substitute

$$(\mathrm{d}x, \mathrm{d}\theta, \mathrm{d}\lambda_+) \mapsto (A_\mathrm{b}, A_\mathrm{f}, A_+).$$

These field redefinitions modify the action in the following way

$$S = S[(\mathrm{d}x, \mathrm{d}\theta, \mathrm{d}\lambda_+) \ \mapsto \ (A_\mathrm{b}, A_\mathrm{f}, A_+)] + \int_\Sigma \big(\tilde{x}\mathrm{d}A_\mathrm{b} + \tilde{\theta}\mathrm{d}A_\mathrm{f} + \sqrt{\alpha}\,\tilde{\lambda}_+\mathrm{d}A_+\big). \tag{4.16}$$

The auxiliary fields $\{A_b, A_f, A_+\}$ are differential 1-forms and $\{\tilde{x}, \tilde{\theta}, \tilde{\lambda}_+\}$ are Lagrange multipliers ensuring that

$$dA_b = d^2A_b = 0, \ dA_f = d^2A_f = 0, \ dA_{\lambda_+} = d^2A_{\lambda_+} = 0, \tag{4.17}$$

since $\{A_b = dx, A_f = d\theta, A_+ = d\lambda_+\}$ and $d^2 = 0$. As a result (see Part I), when we integrate out the Lagrange multipliers we recover the original σ-model action. To find the T-dualised action $\tilde{S}$ out the differential 1-forms $\{A_b, A_f, A_+\}$ instead. A simplification can be made [36]

$$\mathrm{e}^{-B}\big(A_\mathrm{b}P + A_\mathrm{f}Q + \sqrt{\alpha}\,A_+L_+\big)\mathrm{e}^B \ = \ A'_\mathrm{b}P + A'_\mathrm{f}Q + \sqrt{\alpha}\,A'_+L_+ \tag{4.18}$$

when performing this operation, since the Abelian algebra $\langle P, Q, L_+\rangle$ is invariant under conjugation by e^B. We may thus write (4.18) as

[13] The form of this action in terms of the decomposition of Jcan be found in [36], equation (**??**).

$$A_{\rm b}P + A_{\rm f}Q + \sqrt{\alpha}\,A_+L_+ \;=\; {\rm e}^{B}\big(A'_{\rm b}P + A'_{\rm f}Q + \sqrt{\alpha}\,A'_+L_+\big){\rm e}^{-B}\;. \tag{4.19}$$

Consider the following field redefinitions

$$(A_{\rm b}, A_{\rm f}, A_+) \mapsto (A'_{\rm b}P\ , A'_{\rm f}Q\ , \sqrt{\alpha}A'_+L_+),$$

then $A_b = dx$, $A_f = d\theta$, $A_+ = d\lambda_+$ and (4.15) imply that $A'_b = J_P$, $A'_f = J_Q$, and $A'_+ = J_{L_+}$ [36]. Next, substitute

$$\begin{aligned} A_{\rm b} &= \big[{\rm e}^{B}\big(A'_{\rm b}P + A'_{\rm f}Q + \sqrt{\alpha}\,A'_+L_+\big){\rm e}^{-B}\big]_P\ , \\ A_{\rm f} &= \big[{\rm e}^{B}\big(A'_{\rm b}P + A'_{\rm f}Q + \sqrt{\alpha}\,A'_+L_+\big){\rm e}^{-B}\big]_Q\ , \\ A_+ &= \big[{\rm e}^{B}\big(A'_{\rm b}P + A'_{\rm f}Q + \sqrt{\alpha}\,A'_+L_+\big){\rm e}^{-B}\big]_{L_+}, \end{aligned} \tag{4.20}$$

into (4.16) and integrate out $\{A'_b, A'_f, A'_+\}$ to obtain the T-dualised action $\tilde{S}$. Our next goal is to illustrate that $\tilde{S}$ is the Green-Schwartz model in (4.7), with the new coordinates associated with the new choice of coset representative

$$\tilde{g} \;:=\; {\rm e}^{\tilde{x}K + \tilde{\theta}M^{-1}S + \sqrt{\alpha}\,\tilde{\lambda}_+L_-}{\rm e}^{B}{\rm e}^{F(\xi)}\ , \tag{4.21}$$

where $M := (QS)$ and e^B is given in (4.13). The term $F(\xi)$ depends on the background under consideration. For AdS_5 it is given by

$$F(\xi) \;\sim\; -\big[\xi + \xi^5\big]Q + \big[\xi^3 + \xi^7\big]S\ . \tag{4.22}$$

For the AdS_2 cases, we have $F(\xi) = \xi Q$ and for AdS_3, we have that $F(\xi) = \xi Q + \xi^3 S$. Due to the presence of $F(\xi)$ in (4.21), the $\tilde{J} = \tilde{g}^{-1}d\tilde{g}$ arising from (4.21) will generally not contain quadratic terms of fermionic coordinates ξ. Finally, through further complicated field redefinitions $(x,\theta) \to (x',\theta')$ we can bring the dual coset element (4.21) into the form of (4.13)

$$\tilde{g} \;=\; {\rm e}^{\tilde{x}'K + \tilde{\theta}'M^{-1}S + \sqrt{\alpha}\,\tilde{\lambda}'_+L_-}{\rm e}^{B'}{\rm e}^{-\xi' Q}\,.$$

4.4 Example: $AdS_5 \times S^5$ Self-duality

The $AdS_5 \times S^5$ superstring is probably the most important example in the thesis outside of the less than maximal supersymmetric cases, which are the focus of this thesis. In this section we show that $AdS_5 \times S^5$ is self-dual in a κ-covariant way. That is, we have considered all fermionic directions. This result is exciting because it proves that $AdS_5 \times S^5$ is *exactly* self-dual.

4.4.1 Supercoset Action

The $AdS_5 \times S^5$ superspace is accompanied by the following coset

$$\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$$

This coset describes the full type IIB $AdS_5 \times S^5$ background, which is parametrized by ten bosonic coordinates $(X^M) = (x^m, |y|, y^{\hat{a}})$ with $m, n, ... = 0, ..., 3$ and $\hat{a}, \hat{b}, ... = 5, ..., 9$. Additionally there are two 16-component Majorana-Weyl spinor coordinates $\Theta^i = \frac{1}{2}(1 + \Gamma^{11})\Theta^i$ with $i, j, ... = 1, 2$. Both spinors have the same chirality. With this parametrization, we write the line element for $AdS_5 \times S^5$ as

$$ds^2 \;=\; \frac{1}{|y|^2}(dx^m dx^n \eta_{mn} + dy^{\hat{a}} dy^{\hat{a}} + d\hat{y} d\hat{y}) \,, \tag{4.23}$$

where $|y|^2 = (y^{\hat{a}} y^{\hat{a}} + \hat{y}\hat{y})$. The maximally supersymmetric background is supported by the following non-vanishing 5-form components

$$F_{01234} \;=\; -F_{56789} \;=\; 4, \tag{4.24}$$

or, equivalently

$$F_5 \;=\; 4(1 + *)\,\mathrm{Vol}_{\mathrm{AdS}_5} \;=\; 4\,(1 + *)\, e^0 \wedge \cdots \wedge e^4 \tag{4.25}$$

and a dilaton that we set to zero for simplicity.

Lie Superalgebra and Cartan Forms

The general form of a Lie superalgebra for a symmetric space was given in [121]. Inserting (4.25) into this algebra yields the following form of the $\mathfrak{psu}(2,2|4)$ superalgebra

$$\begin{aligned}
&= \eta_{AC} M_{BD} - \eta_{AD} M_{BC} - \eta_{BC} M_{AD} + \eta_{BD} M_{AC} \;, \\
[P_A, P_B] \;&=\; -\tfrac{1}{2} R_{AB}{}^{CD} M_{CD} \;, \\
[M_{AB}, P_C] \;=\; \eta_{AC} P_B - \eta_{BC} P_A \,, \qquad [M_{AB}, \mathcal{Q}_{\alpha i}] \;&=\; -\tfrac{1}{2}(\mathcal{Q}\Gamma_{AB})_{\alpha i} \;, \\
[P_A, \mathcal{Q}_{\alpha i}] \;&=\; -\tfrac{1}{2}(\mathcal{Q}\varepsilon\Gamma^{01234}\Gamma_A)_{\alpha i} \;, \\
\{\mathcal{Q}_{\alpha i}, \mathcal{Q}_{\beta j}\} \;&=\; \mathrm{i}\delta_{ij}(\Gamma^A)_{\alpha\beta}\, P_A - \tfrac{\mathrm{i}}{2}\varepsilon_{ij}(\Gamma^A \Gamma^{01234} \Gamma^B)_{\alpha\beta}\, M_{AB},
\end{aligned} \tag{4.26}$$

where $\varepsilon^{ij} = \varepsilon_{ji}$ and $\varepsilon^{12} = 1$. $(M_{AB}) = (M_{ab}, M_{\hat{a}\hat{b}})$ with $a, b, ... = 0, ..., 4$ and $\hat{a}, \hat{b}, ... = 5, ...9$ generate the $SO(1,4) \times SO(5)$ rotations, whilst $(P_A) = (\mathcal{P}_a, P_{\hat{a}})$ generate the $AdS_5 \times S^5$ translations. The curvature of

AdS_5 and S^5 are given by $R_{ab}^{cd} = 2\delta_{[a}^{c}\delta_{b]}^{d}$ and $R_{\hat{a}\hat{b}}^{\hat{c}\hat{d}} = -2\delta_{[\hat{a}}^{\hat{c}}\delta_{\hat{b}]}^{\hat{d}}$, respectively [36]. The $\mathcal{Q}_{\alpha i}$ are supercharges. For details on the superconformal form of the Lie superalgebra consult [36]. Finally, the corresponding Maurer-Cartan form is

$$J(X,\Theta) \;=\; \tfrac{1}{2}\Omega^{AB}M_{AB} + E^{A}P_{A} + E^{\alpha i}\mathcal{Q}_{\alpha i}, \tag{4.27}$$

which is made from the super-connection $\Omega^{AB}(X,\Theta)$and the supervielbeins $E^{A}(X,\Theta)$ and $E^{\alpha i}(X,\Theta)$. The $\mathfrak{psu}(2,2|4)$ Lie superalgebra in superconformal form (4.10), (4.11) and (4.12) can be written down by defining new bosonic and fermionic generators. The bosonic generators are ($a = m, 4$ with $m = 0,..,3$)

$$\begin{gathered} D \;:=\; \mathcal{P}_4\ , \quad P_m \;:=\; \mathcal{P}_m + M_{m4}\ , \quad K_m \;:=\; -\mathcal{P}_m + M_{m4}\ , \\ M_{mn}\ , \quad R_{\hat{a}} \;:=\; P_{\hat{a}}\ , \quad R_{\hat{a}\hat{b}} \;:=\; -M_{\hat{a}\hat{b}}. \end{gathered} \tag{4.28}$$

The fermionic generators are now defined by

$$\begin{aligned} Q \;&:=\; -\tfrac{1}{\sqrt{2}}(\mathcal{Q}^1 - i\mathcal{Q}^2)\mathbb{P}_+\ , \quad & \hat{Q} \;&:=\; -\tfrac{1}{\sqrt{2}}(\mathcal{Q}^1 + i\mathcal{Q}^2)\mathbb{P}_-\ , \\ S \;&:=\; \tfrac{1}{\sqrt{2}}(\mathcal{Q}^1 + i\mathcal{Q}^2)\mathbb{P}_+\ , \quad & \hat{S} \;&:=\; \tfrac{1}{\sqrt{2}}(\mathcal{Q}^1 - i\mathcal{Q}^2)\mathbb{P}_-\ , \end{aligned} \tag{4.29}$$

with projection operators given by

$$\mathbb{P}_\pm \;:=\; \tfrac{1}{2}(1 \pm \mathrm{i}\Gamma^{0123}).$$

The commutation relations for these generators can be found in [36]. The corresponding basis provides the following non-zero components of the invariant form on $\mathfrak{psu}(2,2|4)$

$$\mathrm{Str}(K_n P_m) \;=\; -2\eta_{mn}\ , \quad \mathrm{Str}(DD) \;=\; 1\ , \quad \mathrm{Str}(S_\alpha Q_\beta) \;=\; 2i(\Gamma^4\mathbb{P}_+)_{\alpha\beta}.$$

Currents and the Supercoset Action

Given the parametrization we have chosen, the coset representative (4.13) becomes

$$g \;:=\; \mathrm{e}^{x^m P_m + \theta^\alpha Q_\alpha}\mathrm{e}^{B}\mathrm{e}^{\xi^\alpha S_\alpha}\ , \quad \mathrm{e}^{B} \;:=\; \mathrm{e}^{\hat{\theta}^\alpha \hat{Q}_\alpha + \hat{\xi}^\alpha \hat{S}_\alpha}|y|^{D}\mathrm{e}^{y^{\hat{a}}R_{\hat{a}}/|y|}. \tag{4.30}$$

As a consequence of the definitions in (4.29), the fermionic variables satisfy the following projection relations $\mathbb{P}_+\theta = \theta$, $\mathbb{P}_+\xi = \xi$, $\mathbb{P}_-\hat{\theta} = \hat{\theta}$ and $\mathbb{P}_-\hat{\xi} = \hat{\xi}$. Following the general setup in Section 4.3, we can derive the form of the Lie superalgebra currents explicitly [36]. Using the commutation relations in [36], we can obtain the components of the current (4.14). Those independent of ξ are

$$
\begin{aligned}
J_{P_m} &= \left[\mathrm{e}^{-B}(\mathrm{d}x^n P_n + \mathrm{d}\theta Q)\mathrm{e}^B\right]_{P_m}\,, \quad J_{Q_\alpha} = \left[\mathrm{e}^{-B}(\mathrm{d}x^n P_n + \mathrm{d}\theta Q)\mathrm{e}^B\right]_{Q_\alpha}\,,\\
J_{\hat{S}_\beta} &= \left[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^B\right]_{\hat{S}_\beta}.
\end{aligned}
\tag{4.31}
$$

The components depending linearly on ξ are given by

$$
\begin{aligned}
J_{K_m} &= -i(\Gamma^m\xi)_\alpha J_{\hat{S}_\alpha}\,, \quad J_D = J_D^{(0)} - i(\Gamma^4\xi)_\alpha J_{Q_\alpha}\,, \quad J_{R_{\hat{a}}} = J_{R_{\hat{a}}}^{(0)} - i(\Gamma^{\hat{a}}\xi)_\alpha J_{Q_\alpha}\,,\\
J_{\hat{Q}} &= J_{\hat{Q}}^{(0)} + (\Gamma_{m4}\xi)^\alpha J_{P_m}\,,\\
J_{M_{mn}} &= J_{M_{mn}}^{(0)} - \tfrac{i}{2}(\xi\Gamma^{mn}\Gamma_4)_\alpha J_{Q_\alpha}\,, \quad J_{R_{\hat{a}\hat{b}}} = J_{R^{(0)}_{\hat{a}\hat{b}}} - \tfrac{i}{2}(\xi\Gamma^{\hat{a}\hat{b}}\Gamma^4)_\alpha J_{Q_\alpha}\,,
\end{aligned}
\tag{4.32}
$$

where the label (0) indicates those terms independent of ξ (i.e. for which $\xi = 0$). Finally there is a current which depends quadratically on ξ

$$
\begin{aligned}
J_{S_\alpha} &= \mathrm{d}\xi^\alpha - \tfrac{1}{2}\xi^\alpha J_D^{(0)} - \tfrac{1}{2}(\Gamma_{mn}\xi)^\alpha J_{M_{mn}}^{(0)} + \tfrac{1}{2}(\Gamma_{\hat{a}4}\xi)^\alpha J_{R_{\hat{a}}}^{(0)} + \tfrac{1}{2}(\Gamma_{\hat{a}\hat{b}\xi)^\alpha} J_{R^{(0)}_{\hat{a}\hat{b}}} + \mathcal{S}^\alpha{}_\beta J_{Q_\beta}\\
&=: J_{S_\alpha}^{(1)} + \mathcal{S}^\alpha{}_\beta J_{Q_\beta},
\end{aligned}
\tag{4.33}
$$

where we define

$$
\mathcal{S}^\alpha{}_\beta := \tfrac{i}{4}\xi^\alpha(\xi\Gamma^4)_\beta + \tfrac{i}{4}(\Gamma^4\Gamma_{\hat{a}}\xi)^\alpha(\xi\Gamma^{\hat{a}})_\beta + \tfrac{i}{8}(\Gamma_{mn}\xi)^\alpha(\xi\Gamma^{mn}\Gamma_4)_\beta - \tfrac{i}{8}(\Gamma^\alpha_{\hat{a}\hat{b}\xi)}(\xi\Gamma^{\hat{a}\hat{b}}\Gamma^4)_\beta, \tag{4.34}
$$

for which $\mathcal{S}^T = -\Gamma^4\mathcal{S}\Gamma^4$. Comparing the Maurer-Cartan current (4.27) with the coset expression in [36] and utilizing the definition of the superconformal generators in (4.28) and (4.29) in terms of the 10-dimensional ones, we are able to write down the relation between the 10-dimensional geometric objects and the components of the supercoset current J. Explicitly,

$$
\begin{aligned}
E^m &= J_{P_m} - J_{K_m}\,, \quad E^4 = J_D\,, \quad E^{\hat{a}} = J_{R_{\hat{a}}},\\
\Omega^{mn} &= 2J_{M_{mn}}\,, \quad \Omega^{m4} = J_{P_m} + J_{K_m}\,, \quad \Omega^{\hat{a}\hat{b}} = -2J_{R_{\hat{a}\hat{b}}}
\end{aligned}
\tag{4.35}
$$

and

$$
E^1 = \tfrac{1}{\sqrt{2}}(J_S + J_{\hat{S}} - J_Q - J_{\hat{Q}})\,, \quad E^2 = \tfrac{i}{\sqrt{2}}(J_S - J_{\hat{S}} + J_Q - J_{\hat{Q}}). \tag{4.36}
$$

Since $E^1 = J_{(1)}$ and $E^2 = J_{(3)}$, E^1 and E^2 have $\mathbb{Z}_4$-gradings 1 and 3, respectively. Then, using (4.31) - (4.33) and the Lagrangians from Section 4.3 for the $PSU(2,2|4)$ supercoset model, the resulting Lagrangian takes the following form

$$
\begin{aligned}
\mathcal{L} &= \tfrac{1}{2}{*E^A} \wedge E^B \eta_{AB} - iE^1 \wedge \Gamma^{01234} E^2 \\
&= \tfrac{1}{2}{*(J_{P_m} - J_{K_m})} \wedge (J_{P_n} - J_{K_n})\eta_{mn} + \tfrac{1}{2}{*J_D} \wedge J_D + \tfrac{1}{2}{*J_{R_{\hat a}}} \wedge J_{R_{\hat a}} - \\
&\quad - \tfrac{i}{2} J_S \wedge \Gamma^4 J_S - \tfrac{i}{2} J_{\hat S} \wedge \Gamma^4 J_{\hat S} + \tfrac{i}{2} J_Q \wedge \Gamma^4 J_Q + \tfrac{i}{2} J_{\hat Q} \wedge \Gamma^4 J_{\hat Q}.
\end{aligned}
\tag{4.37}
$$

Note that we have used (4.36) and the projection properties belonging to the generators (4.29). The NSNS 2-form $B_2 = iE^1 \wedge \Gamma^{01234} E^2$ was calculated from the type IIB supergravity constraints associated with the background (4.25).

4.4.2 T-duality Transformations

Our goal in this section is to T-dualise along x^m and θ^α, that is, all the bosonic and fermionic directions. To do so, we follow the steps detailed in Section 4.3 and then introduce the auxiliary 1-form fields A'^m and A'^α given in (4.18). Using this information together with the dual variables $\tilde x^m$ and $\tilde\theta^\alpha$ the Lagrangian takes the following form

$$
\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 \tag{4.38}
$$

where

$$
\begin{aligned}
\mathcal{L}_1 &:= \tfrac{1}{2}{*A'_m} \wedge A'_n \eta^{mn} - \tfrac{1}{2} A'^m \wedge A'^n \mathcal{M}_{mn} + A'^m \wedge \mathcal{J}_m, \\
\mathcal{L}_2 &:= \tfrac{\mathrm{i}}{2} A'^\alpha \wedge A'^\beta N_{\alpha\beta} - \tfrac{\mathrm{i}}{2}{*A'^\alpha} \wedge A'^\beta (NL)_{\alpha\beta} + A'^\alpha \wedge \mathcal{J}_\alpha, \\
\mathcal{L}_3 &:= \tfrac{1}{2}{*J_{K_m}} \wedge J_{K_n}\eta_{mn} + \tfrac{1}{2}{*J_D^{(0)}} \wedge J_D^{(0)} + \tfrac{1}{2}{*J_{R_{\hat a}}^{(0)}} \wedge J_{R_{\hat a}}^{(0)} - \\
&\quad - \tfrac{\mathrm{i}}{2} J_S^{(1)} \wedge \Gamma^4 J_S^{(1)} - \tfrac{\mathrm{i}}{2} J_{\hat S} \wedge \Gamma^4 J_{\hat S} + \tfrac{\mathrm{i}}{2} J_{\hat Q}^{(0)} \wedge \Gamma^4 J_{\hat Q}^{(0)}.
\end{aligned}
\tag{4.39}
$$

We also have

$$
\begin{aligned}
\mathcal{J}_m &:= -\mathrm{d}\tilde x^n \big[\mathrm{e}^B P_m \mathrm{e}^{-B}\big]_{P_n} - \mathrm{d}\tilde\theta_\alpha \big[\mathrm{e}^B P_m \mathrm{e}^{-B}\big]_{Q_\alpha} + iJ_{\hat Q}^{(0)}\Gamma_m \xi + {*J_{K_m}}, \\
\mathcal{J}_\alpha &:= -\mathrm{d}\tilde x^m \big[\mathrm{e}^B Q_\alpha \mathrm{e}^{-B}\big]_{P_m} + \mathrm{d}\tilde\theta_\beta \big[\mathrm{e}^B Q_\alpha \mathrm{e}^{-B}\big]_{Q_\beta} - \\
&\quad - i{*J_D^{(0)}}(\Gamma^4\xi)_\alpha - i * J_{R_{\hat a}}^{(0)}(\Gamma^{\hat a}\xi)_\alpha - i(J_S^{(1)}\Gamma^4\mathcal{S})_\alpha
\end{aligned}
\tag{4.40}
$$

and

$$
\begin{aligned}
\mathcal{M}_{AB} &:= i\xi\Gamma_{AB}\Gamma_4\xi \ , \\
N_{\alpha\beta} &:= \big(\Gamma^4(1+\mathcal{S}^2)\big)_{\alpha\beta} \ , \quad (NL)_{\alpha\beta} := i(\Gamma^4\xi)_\alpha(\Gamma^4\xi)_\beta + i(\Gamma^{\hat a}\xi)_\alpha(\Gamma_{\hat a}\xi)_\beta.
\end{aligned}
\tag{4.41}
$$

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_1 + \tilde{\mathcal{L}}_2 + \mathcal{L}_3 \tag{4.42}$$

with

$$\begin{aligned} \tilde{\mathcal{L}}_1 &:= \tfrac{1}{2}{*\mathcal{J}^m} \wedge \mathcal{J}^n[(\eta - \mathcal{M}^2)^{-1}]_{mn} + \tfrac{1}{2}\mathcal{J}^m \wedge \mathcal{J}^n[\mathcal{M}(1-\mathcal{M}^2)^{-1}]_{mn}, \\ \tilde{\mathcal{L}}_2 &:= \tfrac{\mathrm{i}}{2}\mathcal{J} \wedge (N - NL^2)^{-1}\mathcal{J} + \tfrac{\mathrm{i}}{2}{*\mathcal{J}} \wedge L(N-NL^2)^{-1}\mathcal{J}. \end{aligned} \tag{4.43}$$

and $\mathcal{L}_3$ is the same as before[14]. We can simplify $\tilde{\mathcal{L}}_1$, then we may write it as (see [36] for details)

$$\tilde{\mathcal{L}}_1 = \tfrac{1}{2}(1 - \tfrac{1}{4}\mathcal{M}_{\hat{a}\hat{b}}\mathcal{M}^{\hat{a}\hat{b}})^{-1}\big({*\mathcal{J}^m} \wedge \mathcal{J}^n \eta_{mn} + \mathcal{J}^m \wedge \mathcal{J}^n \mathcal{M}_{mn}\big)\,.$$

To complete the self-dualisation process, we would like to cast the dual Lagrangian in the form of the original Lagrangian[15]. The result of this action are the new expressions for the currents $\mathcal{J}_m$ and $\mathcal{J}_\alpha$

$$\begin{aligned} \mathcal{J}_m &= \tilde{J}^{(1)}_{P_m} - \tilde{J}^{(0)}_{K_m} + {*J_{K_m}}\ , \\ \mathcal{J}_\alpha &= \mathrm{i}(\Gamma^4 \tilde{J}^{(0)}_S)_\alpha - \mathrm{i}J^{(1)}_{S_\beta}\Gamma^4_{\beta\gamma}\mathcal{S}^\gamma{}_\alpha - \mathrm{i}{*J^{(0)}_D}(\Gamma^4\xi)_\alpha - \mathrm{i}{*J^{(0)}_{R_{\hat{a}}}}(\Gamma_{\hat{a}}\xi)_\alpha\,. \end{aligned} \tag{4.44}$$

The new (dual) choice of coset representative which leads to the Maurer-Cartan forms[16] is given by

$$\tilde{J} = \tilde{g}^{-1}\mathrm{d}\tilde{g}\ , \quad \tilde{g} := \mathrm{e}^{\tilde{x}^n K_n - \mathrm{i}\tilde{\theta}\Gamma^4 S}\mathrm{e}^{B}\mathrm{e}^{-(\xi Q + S_\alpha \mathcal{S}^\alpha{}_\beta \xi^\beta)(1-\frac{1}{4}\mathcal{M}_{\hat{a}\hat{b}}\mathcal{M}^{\hat{a}\hat{b}})^{-1}}\ , \tag{4.45}$$

Our goal is to show that $\mathcal{L}_g = \mathcal{L}_{\tilde{g}}$, therefore we need to show that $g \sim \tilde{g}$. Through a complicated change of variables [36], the dual coset element (4.45) can be brought into the same form as the original coset element (4.30). This means that the last exponent to the right takes the form[17] $e^{-\xi' Q}$, similar to g. The dual coset element becomes

$$\tilde{g} \approxeq \mathrm{e}^{\tilde{x}'^n K_n - \mathrm{i}\tilde{\theta}'\Gamma^4 S}\mathrm{e}^{B'}e^{-\xi' Q}, \tag{4.46}$$

where $e^{B'} = \mathrm{e}^{B}\mathrm{e}^{-\frac{\mathrm{i}}{2}D(\xi\Gamma^4\mathcal{S}\xi + \mathcal{O}(\xi^8)) - \frac{\mathrm{i}}{2}R_{\hat{a}}\,\xi\Gamma^{\hat{a}}\mathcal{S}\xi}$, $\tilde{x}'^n = \tilde{x}^n + f^n(y,\hat{\theta},\hat{\xi},\xi)$, $\tilde{\theta}'_\alpha = \tilde{\theta}_\alpha + f_\alpha(y,\hat{\theta},\hat{\xi},\xi)$ and f^n and f_α are functions of the coordinates $y^{\hat{\alpha}}$ of S^5, the radial direction $|y|$ of AdS_5 and the Grassmann-odd coordinates $\hat{\theta}$, $\hat{\xi}$, and ξ. The choices (4.45) and (4.46) have associated $\mathbb{Z}_4$ - automorphism

[14] $\mathcal{L}$ is a function of currents not involved in the T-dualisation process and therefore remains invariant under dualisation.

[15] For details see [36].

[16] Found in (3.21) in [36].

[17] Details may be found in [36].

$$\begin{aligned} P_m &\leftrightarrow K_m\,, \quad D \to -D, \quad R_{\hat{a}} \to -R_{\hat{a}}, \\ S \to -\mathrm{i}Q\,, &\quad \hat{S} \to -\mathrm{i}\hat{Q}, \quad Q \to -\mathrm{i}S\,, \quad \hat{Q} \to -\mathrm{i}\hat{S} \end{aligned} \tag{4.47}$$

of the $\mathfrak{p}su(2,2|4)$ Lie superalgebra. The choice (4.45) of the dual element $\tilde{g}$, tells us that the ξ-independent components $\tilde{J}^{(0)}_{Q}$, $\tilde{J}^{(0)}_{\hat{S}}$, $\tilde{J}^{(0)}_{D}$, $\tilde{J}^{(0)}_{R_{\hat{a}}}$, $\tilde{J}^{(0)}_{R_{\hat{a}\hat{b}}}$ and $\tilde{J}^{(0)}_{M_{mn}}$ of the currents are the same as the ones without tilde, whereas the full expressions for the dual currents $\tilde{J}_m$, $\tilde{J}_{K_m}$, $\tilde{J}_{\hat{S}}$, and $\tilde{J}_{\hat{Q}}$ are

$$\begin{aligned} \tilde{J}_{P_m} &= \tilde{J}^{(1)}_{P_m}\big(1 - \tfrac{1}{4}\mathcal{M}_{\hat{a}\hat{b}}\mathcal{M}^{\hat{a}\hat{b}}\big)^{-\frac{1}{2}}\,, \\ \tilde{J}_{K_m} &= (\tilde{J}^{(0)}_{K_m} - J_{K_l}\mathcal{M}_l{}^m)\big(1 - \tfrac{1}{4}\mathcal{M}_{\hat{a}\hat{b}}\mathcal{M}^{\hat{a}\hat{b}}\big)^{-\frac{1}{2}}, \\ \tilde{J}_{\hat{S}} &= \tilde{J}^{(0)}_{\hat{S}} + \tilde{J}_{K_m}(\Gamma_4\Gamma_m\xi) + \frac{1}{2}J_{K_n}\mathcal{M}_{nm}(\Gamma^4\Gamma^m\xi)\big(1 + \tfrac{1}{16}\mathcal{M}_{\hat{a}\hat{b}}\mathcal{M}^{\hat{a}\hat{b}}\big)\,, \\ \tilde{J}_{\hat{Q}} &= \tilde{J}^{(0)}_{\hat{Q}} + \tfrac{1}{2}\tilde{J}^{(1)}_{P_m}(\Gamma^4\Gamma^n\xi)\mathcal{M}_{nm}\big(1 + \tfrac{3}{16}\mathcal{M}_{\hat{a}\hat{b}}\mathcal{M}^{\hat{a}\hat{b}}\big). \end{aligned} \tag{4.48}$$

Similarly, the dual currents $\tilde{J}_{R_{\underline{a}}} = (\tilde{J}_D\,,\,\tilde{J}_{R_{\hat{a}}})$, $\tilde{J}_Q$ and $\tilde{J}_S$ are

$$\begin{aligned} \tilde{J}_{R_{\underline{a}}} &= \tilde{J}^{(0)}_{R_{\underline{a}}} - \mathrm{i}(\tilde{J}^{(0)}_S\Gamma^4 - J^{(1)}_S\Gamma^4\mathcal{S})(N - NL^2)^{-1}\Gamma^{\underline{a}}\xi + \tfrac{\mathrm{i}}{2}J^{(0)}_{R_{\underline{b}}}\xi\Gamma^{\underline{a}}L(N - NL^2)^{-1}\Gamma_{\underline{b}}\xi \\ &\quad - \tfrac{1}{2}(\tilde{J}^{(0)}_S + \mathcal{S}J^{(1)}_S)\Gamma_{\underline{b}}\xi(\xi\Gamma^{\underline{b}}L\Gamma^4\Gamma^{\underline{a}}\xi) + \tfrac{1}{8}J^{(0)}_{R_{\underline{c}}}\xi\Gamma^{\underline{b}}L\Gamma^4\Gamma^{\underline{a}}\xi(\xi\Gamma_{\underline{b}}L\Gamma^4\Gamma_{\underline{c}}\xi), \\ \tilde{J}_Q &= \big[N^2(1 - L^2)\big]^{-\frac{1}{2}}(\tilde{J}^{(1)}_Q - \mathcal{S}\tilde{J}^{(0)}_S)\,, \quad \tilde{J}^{(1)}_Q = -J^{(1)}_S - J^{(0)}_{R_{\underline{a}}}(\Gamma^4\Gamma_{\underline{a}}\xi), \\ \tilde{J}_S &= \big[N^2(1 - L^2)\big]^{-\frac{1}{2}}\big[\tilde{J}^{(0)}_S - \mathcal{S}\tilde{J}^{(1)}_Q + (2\mathcal{S} - \Gamma^4 NL)J^{(1)}_S\big]. \end{aligned} \tag{4.49}$$

Then, upon substituting the first two equations in (44.48) into (4.44), we get the following expression for $\mathcal{J}_m$

$$\mathcal{J}^m \;=\; (\tilde{J}_{P_m} - \tilde{J}_{K_m})(1 - \tfrac{1}{4}\mathcal{M}_{\hat{a}\hat{b}}\mathcal{M}^{\hat{a}\hat{b}})^{\frac{1}{2}} + *J_{K_m} - J_{K_n}\mathcal{M}^m_n. \tag{4.50}$$

Using (4.48) and (4.49), we find the relations

$$-\tfrac{\mathrm{i}}{2}\tilde{J}_{\hat{S}}\Gamma^4\tilde{J}_{\hat{S}} + \tfrac{\mathrm{i}}{2}\tilde{J}^{(0)}_{\hat{S}}\Gamma^4\tilde{J}^{(0)}_{\hat{S}} \;=\; \tilde{J}^{(0)}_{K_m}J_{K_n}\eta_{mn} + \tfrac{1}{2}\tilde{J}_{K_m}\tilde{J}_{K_n}\mathcal{M}_{mn} - \tfrac{1}{2}J_{K_m}J_{K_n}\mathcal{M}_{mn} \tag{4.51}$$

and

$$\tfrac{1}{2}\tilde{J}_{P_m}\tilde{J}_{P_n}\mathcal{M}_{mn} \;=\; \tfrac{\mathrm{i}}{2}\tilde{J}_{\hat{Q}}\Gamma^4\tilde{J}_{\hat{Q}} - \tfrac{\mathrm{i}}{2}\tilde{J}^{(0)}_{\hat{Q}}\Gamma^4\tilde{J}^{(0)}_{\hat{Q}}. \tag{4.52}$$

Furthermore, combining (4.50), (4.51) and (4.52) with the Lagrangians $\tilde{\mathcal{L}}_1$ and $\mathcal{L}_3$, it follows that $\tilde{\mathcal{L}} + \mathcal{L}_3$ has the following form

$$\begin{aligned}\tilde{\mathcal{L}}_1+\mathcal{L}_3 &= \tfrac{1}{2}*(\tilde{J}_{P_m}-\tilde{J}_{K_m})\wedge(\tilde{J}_{P_n}-\tilde{J}_{K_n})\eta_{mn}-\tfrac{\mathrm{i}}{2}\tilde{J}_{\hat{S}}\wedge\Gamma^4\tilde{J}_{\hat{S}}+\tfrac{\mathrm{i}}{2}\tilde{J}_{\hat{Q}}\wedge\Gamma^4\tilde{J}_{\hat{Q}}+\\ &+\tfrac{1}{2}*J_D^{(0)}\wedge J_D^{(0)}+\tfrac{1}{2}*J_{R_{\hat{a}}}^{(0)}\wedge J_{R_{\hat{a}}}^{(0)}-\tfrac{\mathrm{i}}{2}J_S^{(1)}\wedge\Gamma^4 J_S^{(1)}-\\ &-\tilde{J}_{K_m}\wedge\tilde{J}_{P_n}\mathcal{M}_{mn}-\tilde{J}_{P_m}^{(1)}\wedge J_{K_n}\eta_{mn}\ .\end{aligned}\tag{4.53}$$

Additionally, using the first equation in (4.49), we find that the Lagrangian $\tilde{\mathcal{L}}_2$ becomes

$$\begin{aligned}\tilde{\mathcal{L}}_2 &= \tfrac{1}{2}*\tilde{J}_{R_{\underline{a}}}\wedge\tilde{J}_{R_{\underline{a}}}-\tfrac{1}{2}*J_{R_{\underline{a}}}^{(0)}\wedge J_{R_{\underline{a}}}^{(0)}-\\ &-\tfrac{\mathrm{i}}{2}\big(\tilde{J}_S^{(0)}+\mathcal{S}J_S^{(1)}\big)\wedge(N-NL^2)^{-1}\big(\tilde{J}_S^{(0)}+\mathcal{S}J_S^{(1)}\big)\\ &+\mathrm{i}J_{R_{\underline{a}}}^{(0)}\wedge\big(\tilde{J}_S^{(0)}+\mathcal{S}J_S^{(1)}\big)\Gamma^4 L(N-NL^2)^{-1}\Gamma_{\underline{a}}\xi+\\ &+\tfrac{\mathrm{i}}{2}J_{R_{\underline{a}}}^{(0)}\wedge J_{R_{\underline{b}}}^{(0)}\xi\Gamma_{\underline{a}}(N-NL^2)^{-1}\Gamma_{\underline{b}}\xi\ .\end{aligned}\tag{4.54}$$

Using the above equations (4.53) and (4.54), we obtain

$$\tilde{\mathcal{L}} \;=\; \tilde{\mathcal{L}}_1+\tilde{\mathcal{L}}_2+\mathcal{L}_3=\mathcal{L}_{\tilde{g}}+\mathcal{L}'+\mathcal{L}'',\tag{4.55}$$

where

$$\begin{aligned}\mathcal{L}_{\tilde{g}} &:= \tfrac{1}{2}*(\tilde{J}_{P_m}-\tilde{J}_{K_m})\wedge(\tilde{J}_{P_n}-\tilde{J}_{K_n})\eta_{mn}+\tfrac{1}{2}*\tilde{J}_D\wedge\tilde{J}_D+\tfrac{1}{2}*\tilde{J}_{R_{\hat{a}}}\wedge\tilde{J}_{R_{\hat{a}}}\\ &-\tfrac{\mathrm{i}}{2}\tilde{J}_S\wedge\Gamma^4\tilde{J}_S-\tfrac{\mathrm{i}}{2}\tilde{J}_{\hat{S}}\wedge\Gamma^4\tilde{J}_{\hat{S}}+\tfrac{\mathrm{i}}{2}\tilde{J}_Q\wedge\Gamma^4\tilde{J}_Q+\tfrac{\mathrm{i}}{2}\tilde{J}_{\hat{Q}}\wedge\Gamma^4\tilde{J}_{\hat{Q}}\end{aligned}\tag{4.56}$$

is constructed in terms of the $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ currents built from the dual coset element. Whilst, $\mathcal{L}'$ and $\mathcal{L}''$ are given by

$$\begin{aligned}\mathcal{L}' &:= \mathrm{i}J_S^{(1)}\wedge(N-NL^2)^{-1}N^{-1}\Gamma^4\tilde{J}_Q^{(1)}-\mathrm{i}J_S^{(1)}\wedge\Gamma^4\mathcal{S}(N-NL^2)^{-1}\mathcal{S}\tilde{J}_Q^{(1)}+\\ &+J_{K_m}\wedge\tilde{J}_{P_n}^{(1)}\eta_{mn}(1+\tfrac{1}{4}\mathcal{M}_{\hat{a}\hat{b}}\mathcal{M}^{\hat{a}\hat{b}})\end{aligned}\tag{4.57}$$

and

$$\begin{aligned}\mathcal{L}'' &:= \mathrm{i}\tilde{J}_S^{(0)}\wedge(N-NL^2)^{-1}(4N^{-1}\Gamma^4\mathcal{S}-\mathcal{S}-L)J_S^{(1)}-\\ &-\mathrm{i}J_{R_{\underline{a}}}^{(0)}\wedge\tilde{J}_S^{(0)}(N-NL^2)^{-1}N^{-1}\Gamma^4(2\mathcal{S}-\Gamma^4 NL)\Gamma^4\Gamma_{\underline{a}}\xi-\\ &-\tilde{J}_{K_m}^{(0)}\wedge\tilde{J}_{P_n}^{(1)}\mathcal{M}_{mn}\big(1+\tfrac{1}{4}\mathcal{M}_{\hat{a}\hat{b}}\mathcal{M}^{\hat{a}\hat{b}}\big).\end{aligned}\tag{4.58}$$

Realising that (4.56) has the same explicit form as the initial Lagrangian (4.38), hence the Lagrangian of the superstring on $AdS_5\times S^5$ will be self-dual provided that $\mathcal{L}'+\mathcal{L}''$ is a total derivative. This can be demonstrated by performing some involved computations[18]. The results are

[18] See [36] for details.

$$\mathcal{L}' \;=\; -\mathrm{i}\,\mathrm{d}\Big[J^{(1)}_{S}\Gamma^4\xi\big(1+\tfrac{1}{4}\mathcal{M}_{\hat{a}\hat{b}}\mathcal{M}^{\hat{a}\hat{b}}\big)\Big], \tag{4.59}$$

and

$$\mathcal{L}'' \;=\; -\tfrac{1}{2}\mathrm{d}\Big[\tilde{J}^{(1)}_{M_{mn}}\mathcal{M}_{mn}\big(1+\tfrac{1}{4}\mathcal{M}_{\hat{a}\hat{b}}\mathcal{M}^{\hat{a}\hat{b}}\big)\Big], \tag{4.60}$$

where the matrices $\mathcal{M}_{mn}$ and $\mathcal{M}_{\hat{a}\hat{b}}$ have been defined by the first equation in (4.41).

Lastly, we have showed that $\int \mathcal{L}_g = \int \mathcal{L}_{\tilde{g}}$ which demonstrates that $AdS_5 \times S^5$ is exactly self-dual. That is, the $AdS_5 \times S^5$ superstring action is self-dual under the worldsheet duality transformation on (x^m, θ^α) coordinates, without gauge fixing κ-symmetry.

4.5 Summary

In this chapter we have introduced the technicalities involved in working with worldsheet T-duality, which contrasts the methods of Chapter 3, involving the supergravity. After an introduction, the ideas involving coset σ-models were slowly introduced. After some useful examples, the general setup for the work that follows was given. Finally, we end with the important example of the self-duality of the $AdS_5 \times S^5$ background. The worldsheet T-duality transforms the superstring σ-model action, which is constructed with the use of the supercoset representative g, into the action constructed using the supercoset element $\tilde{g}$. The duality makes use of a $\mathbb{Z}_4$-automorphism of the $\mathfrak{psu}(2,2|4)$ Lie superalgebra. Whilst the worldsheet approach is challenging as a result of the difficult mathematical tools involved, the worldsheet approach is invaluable as it offers us insight that is missed when working in the supergravity picture. It beautifully demonstrates the T-self-duality exactly, that is, without fixing κ-symmetry gauge.

Chapter 5

Less Than Maximally Supersymmetric Coset Models

5.1 Introduction

This chapter deals with the self-duality of some of the backgrounds considered in Chapter 3 (i.e. $AdS_d \times S^d$ $(d = 2, 3)$ and $AdS_d \times S^d \times S^d$ $(d = 2, 3)$), but it does so from the perspective of the coset geometry. This is also known as the worldsheet perspective. When we consider the string theory, we attribute the existence of dual superconformal symmetry to the self-duality of the chosen superstring σ-model given certain T-duality transformations along bosonic and fermionic string modes on the worldsheet which we associate with (anti-) commuting isometries of the $AdS_5 \times S^5$ background [21, 22]. Self-duality is the direct consequence of performing a sequence of bosonic and fermionic T-duality transformations. Specifically, we say a background is self-dual when these transformations do not change the values of the background fields. In particular the RR fields and the dilaton[1] remain invariant. The relationship between fermionic T-duality and dual superconformal symmetry, in the context of the $AdS_5 \times S^5$ superstring and the corresponding super Yang-Mills theory, is very well understood [21–23]. However, there are cases involving σ-models which possess less than maximal supersymmetry. These backgrounds are also integrable, but are less understood as compared to the maximal case. They will be the focus of this chapter.

We concentrate on the remaining issues regarding the T-duality of superstrings on $AdS_d \times S^d \times M^{10-2d}$ backgrounds. We would like to obtain a better understanding of these cases with the hope that it provides insights into the still problematic $AdS_4 \times \mathbb{C}P^3$ background. However, for some recent developments in this regard, see [42]. The problem stems from issues which appear when performing fermionic T-duality on the associated σ-model [124, 151] and the supergravity background itself [16, 101, 107, 151]. T-self-duality is shown for supercoset σ-models associated with strings propagating on $AdS_d \times S^d$ $(d = 2, 3)$ backgrounds upon imposing a partial gauge fixing of κ-symmetry of the σ-model actions by putting to zero one quarter of the supercoset fermion modes [36]. An additional issue arises when considering the coset supermodels on $AdS_d \times S^d$ $(d = 2, 3)$ backgrounds as a result of the fact that coset supermodels only

[1] See [23] for a review and references.

describe some sectors of the full superstring theory on $AdS_d \times S^d \times M^{10-2d}$ backgrounds [36]. For $d = 3$, only 16 out of 32 supersymmetries in 10-dimensions are preserved by these backgrounds. For $d = 2$, only 8 out of 32 supersymmetries are preserved. This implies that 16 and 8 fermionic modes, respectively, correspond to the fermionic directions of the associated coset superspace, with the remaining 16 and 24 fermionic modes corresponding to broken supersymmetries [36]. These sectors of the theory are coupled non-trivially to the non-supercoset directions of M^{10-2d} via these modes.

For d = 3, we may use κ-symmetry to set all of the 16 non-supercoset fermionic modes to zero. However, this gauge fixing does not work for a large class of classical string solutions (including strings moving on the $AdS_3 \times S^3$ sub-space only [152]). Even though the $AdS_3 \times S^3$ supercoset σ-model with 16 fermions possesses κ-symmetry [36], this κ-symmetry is broken when the supercoset model is coupled to the T^4 sector (via the Virasoro constraints) of the full superstring action in $AdS_3 \times S^3 \times T^4$ [37]. Here, the 16 non-supercoset fermions have been κ gauge fixed to zero [36]. This means that the κ-symmetry of the $AdS_3 \times S^3$ supercoset sub-sector is part of the κ-symmetry of the complete 10-dimensional superstring which is lost when the non-supercoset fermions have been completely gauge fixed. For the $d = 2$ case we may use κ-symmetry to remove 16 of the 24 non-supercoset fermions, then at least 8 of the remaining non-supercoset fermionic modes are always present in the $AdS_2 \times S^2 \times T^6$ string spectrum [37, 39]. Self-duality of the associated supercoset models have been demonstrated for partially gauge fixed κ-symmetry. Here, we have set some of the fermionic supercoset coordinates to zero [122, 124]. However, when supercoset models are used to describe gauge fixed sectors of the superstring σ-model where κ-symmetry has already been used to remove part of the non-supercoset fermions, we may no longer use κ-symmetry to demonstrate the self-duality of the corresponding supercoset sectors of $AdS_d \times S^d \times M^{10-2d}$ superstrings.

Resulting from these issues is an appreciation of the importance of proving the T-self-duality of superstrings on $AdS_d \times S^d \times M^{10-2d}$ backgrounds without gauge fixing κ-symmetry, that is, by taking into account the non-supercoset fermions. This is precisely what we will demonstrate in the following section. Figure 5.1 illustrates an example of the $AdS_2 \times S^2 \times T^6$ case. Furthermore, we consider the exceptional cases, those backgrounds with two sphere subspaces. There we isolate strings movement to the $AdS_d \times S^d \times S^d$ sectors and fix the non-supercoset fermions to zero. We show that these backgrounds are self dual under combined bosonic and fermionic T-duality. This chapter is based on research presented in [36].

5.2 Self-duality of $AdS_3 \times S^3 \times T^4$ Superstrings

Solutions of the type $AdS_3 \times S^3 \times T^4$ preserve 16 supersymmetries. These supersymmetries generate the required superisometries to form the $PSU(1,1|2) \times PSU(1,1|2)$ supergroup (see for example [153]), a subgroup of $PSU(2,2|4)$. The curvature of the $AdS_3 \times S^3$ subspace is given by

$$R^{ab}_{\mathrm{AdS}_3} = -e^a \wedge e^b , \quad R^{\hat{a}\hat{b}}_{S^3} = e^{\hat{a}} \wedge e^{\hat{b}}, \tag{5.1}$$

where $e^a = e^a(x)$ for $a, b, \ldots = 0, 1, 4$ and $e^{\hat{a}} = e^{\hat{a}}(y)$ for $\hat{a}, \hat{b}, \ldots = 5, 6, 7$ are the vielbeins of AdS_3 and S^3, respectively. The radius of curvature for both manifolds have been set to one for convenience.

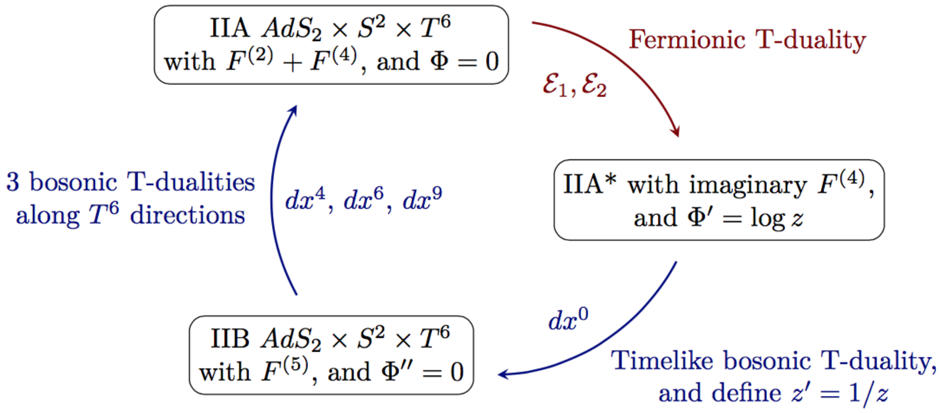

Figure 5.1: The idea of self-duality we study is that a sequence of bosonic and fermionic T-dualities returns us to the same background. This is shown here for the case in which we start with type IIB $AdS_2 \times S^2 \times T^6$ supported by an $F^{(5)}$ Ramond-Ramond flux [36].

Backgrounds of this type may be supported by a 5-form flux given by

$$F_5 \;=\; \tfrac{1}{3}(\varepsilon_{cba} e^a \wedge e^b \wedge e^c + \varepsilon_{\hat{c}\hat{b}\hat{a}} e^{\hat{a}} \wedge e^{\hat{b}} \wedge e^{\hat{c}}) \wedge (\mathrm{d}\varphi^2 \wedge \mathrm{d}\varphi^3 + \mathrm{d}\varphi^8 \wedge \mathrm{d}\varphi^9), \tag{5.2}$$

where $\mathrm{d}\varphi^{a'}$ $(a', b', \ldots = 2, 3, 8, 9)$ are the flat vielbeins along T^4. It is important to note the change in the form of the F_5-flux (5.2) as compared with its value in the $AdS_5 \times S^5$ solution. This difference results in changing the geometry of $D = 10$ spacetime geometry, which breaks half of the 32 supersymmetries. Only half of the maximal supersymmetry is preserved, the fermionic modes of the string in these backgrounds split into 16 fermions ϑ which are associated with the preserved, and because of this, the string fermionic modes in such a background split into 16 fermions ϑ associated with the preserved symmetries and 16 fermions υ associated with the broken symmetries.

Explicitly, the splitting is realized by using the additional projectors $\frac{1}{2}(1 \pm \Gamma^{2389})$ as follows

$$\vartheta^i \;=\; \tfrac{1}{2}(1 - \Gamma^{2389})\Theta^i \;, \quad \upsilon^i \;=\; \tfrac{1}{2}(1 + \Gamma^{2389})\Theta^i. \tag{5.3}$$

As for the $AdS_5 \times S^5$ case, the fermions ϑ can be regarded as Grassmann-odd directions of the supercoset space

$$\frac{PSU(1,1|2) \times PSU(1,1|2)}{SO(1,2) \times SU(2)}$$

which contains $\mathrm{AdS}_3 \times S^3$ as the bosonic subspace. The T^4 directions and the non-supercoset fermions υ are responsible for extending this supercoset to a full solution obeying the 10-dimensional type IIB supergravity constraints. For certain classical string solutions in $AdS_3 \times S^3 \times T^4$, κ-symmetry may be used to gauge fix to zero all the non-supercoset fermions υ. Modulo some constraints (see for example [36]), this gauge permits fluctuations of the string along T^4 to decouple from the $\frac{PSU(1,1|2)\times PSU(1,1|2)}{SO(1,2)\times SU(2)}$ modes (see [37] for details). Hence, the superstring action reduces to its supercoset part, which can be obtained as a truncation of the $AdS_5 \times S^5$ action, once we select the $PSU(1,1|2) \times PSU(1,1|2)$ subgroup of $PSU(2,2|4)$ and reduce to it.

First, we identify the 10-dimensional indices 2, 3, 8, and 9 as associated with the T^4 directions $(\varphi^{a'}) = (\varphi^I, \varphi^{I'})$ $(I, J, \ldots = 2,3;\ I', J', \ldots = 8,9)$. This particular choice of the tangent space indices associated with $AdS_3 \times S^3 \times T^4$ are related to the way we truncate the $\mathfrak{psu}(2,2|4)$ superalgebra to $\mathfrak{psu}(1,1|2) \oplus \mathfrak{psu}(1,1|2)$. Next, we remove from the algebra all the bosonic generators with the indices $m = 2,3$ and $\hat{a} = 8,9$ from the algebra, and halve the number of fermionic generators by acting on the original 32 generators defined in (4.26) with the additional projector introduced in section 4.4

$$\mathcal{Q}^i \ = \ \tfrac{1}{2}\mathcal{Q}^i(1 - \Gamma^{2389})\ , \qquad i \ = \ 1,2. \tag{5.4}$$

The generators $(Q, \hat{Q}, S, \hat{S})$ defined in section 4.4 are subject to the same projection. The algebra $\mathfrak{psu}(1,1|2) \oplus \mathfrak{psu}(1,1|2)$ may be found in [36]. From a geometrical point of view, this truncation corresponds to obtaining the $AdS_3 \times S^3 \times T^4$ background from $AdS_5 \times S^5$ by formally compactifying two directions of the 4-dimensional Minkowski boundary of AdS_5 and two directions of S^5 onto $T^4 \cong T^2 \times T^2$. Finally, one deforms the value of the F_5 flux as in (5.2).

5.2.1 Self-duality of the Supercoset Model

As already mentioned, $\frac{PSU(1,1|2)\times PSU(1,1|2)}{SO(1,2)\times SU(2)}$ can be described as a truncation of the $AdS_5 \times S^5$ supercoset model. Then, it follows that we may use the results of the previous Chapter (section 4.4) to show that $AdS_3 \times S^3 \times T^4$ is self-dual under the combined T-dualisation of the bosonic coordinates (along the 2-dimensional Minkowski boundary of AdS_3) and four fermionic directions (associated with a commuting subalgebra of the $PSU(1,1|2) \times PSU(1,1|2)$ isometries). Conveniently, the $\frac{PSU(1,1|2)\times PSU(1,1|2)}{SO(1,2)\times SU(2)}$ supercoset σ-model Lagrangian has the same form as (4.37) where the currents constructed with the coset element having a form similar to (4.30). Then, if we follow the steps in section 4.3 we find that, after the T-dualisation process, the Lagrangian turns is equal (up to a total derivative) to the $AdS_3 \times S^3$ σ-model Lagrangian constructed in terms of the supercoset element

$$\tilde{J} \ = \ \tilde{g}^{-1}\mathrm{d}\tilde{g}\ , \qquad \tilde{g} \ := \ \mathrm{e}^{\tilde{x}^n K_n - \mathrm{i}\tilde{\theta}\Gamma^4 S}\mathrm{e}^{B}\mathrm{e}^{-(\xi Q + S_\alpha \mathcal{S}^\alpha{}_\beta \xi^\beta)}\ , \tag{5.5}$$

where $\mathcal{S}^\alpha{}_\beta$ was defined in (4.34). For the case $AdS_3 \times S^3$, at most ξ^4 powers are present and thus, the factor $(1 - \frac{1}{4}\mathcal{M}_{\hat{a}\hat{b}}\mathcal{M}^{\hat{a}\hat{b}})^{-1}$ appearing in the $AdS_5 \times S^5$ supercoset element does not enter the expression for $\tilde{g}$ here. Furthermore, we may bring the coset element $\tilde{g}$ in (5.5) into a form similar to that of the original coset element (4.30). Again, this means that we arrive at an expression for $\tilde{g}$ with the last factor to the right being composed of $\mathrm{e}^{-\xi Q}$ alone. This was achieved in the $AdS_5 \times S^5$ case as well. The final form of $\tilde{g}$ (after applying a chain of equalities as in [36]) is

$$\tilde{g} := \mathrm{e}^{\tilde{x}^n K_n - \mathrm{i}\tilde{\theta}\Gamma^4 S}\mathrm{e}^{B'}\mathrm{e}^{-\xi Q} \tag{5.6}$$

where $\mathrm{e}^{B'} = \mathrm{e}^{B}\,\mathrm{e}^{-\frac{i}{2}D\xi\Gamma^4\mathcal{S}\xi}$, $\tilde{x}'^n = \tilde{x}^n + f^n(y,\hat{\theta},\hat{\xi},\xi)$, $\tilde{\theta}'_\alpha = \tilde{\theta}_\alpha + f_\alpha(y,\hat{\theta},\hat{\xi},\xi)$ and f^n and f_α are certain functions of the coordinates $y^{\hat{a}}$ of S^3, the radial direction $|y|$ of AdS_3 and the Grassmann-odd coordinates $\hat{\theta}$, $\hat{\xi}$, and ξ. The $AdS_2 \times S^2$ case follows similarly (see [36] for a detailed presentation). There we have to consider type IIA and IIB backgrounds separately, although both backgrounds are related by a bosonic T-duality.

5.2.2 Non-supercoset Fermions

So far we have demonstrated that, in a particular κ-symmetry gauge in which the 16 non-supercoset fermions υ are set to zero, the $AdS_3 \times S^3 \times T^4$ superstring action is self-dual under the combined fermionic and bosonic T-duality. However, as mentioned before, this gauge is not always permitted. For example, when the classical strings move entirely in the $AdS_3 \times S^3$ subspace. The κ-symmetry projector (M in [152]) fixes the gauge. Meaning that it projects out all but the physical degrees of freedom. Suppose that the κ-symmetry projector commutes with the projector which singles out the unbroken supersymmetries (16 in this case). Then, it cannot eliminate enough non-supercoset fermions to leave us with a physical set of modes. Some unphysical modes will always remain. Consequently, some solutions cannot be found in some gauges and what we have discovered is that the motion of the string in the AdS_3 subspace is important. It is not consistent with our gauge fixing. Thus, it is important to understand how the T-dualisation works in different gauges or without fixing κ-symmetry.

To derive the form of the Green–Schwarz superstring Lagrangian (4.6) in the $AdS_3 \times S^3 \times T^4$ background, we need expressions for the supervielbeins $\mathcal{E}^A(X,\vartheta,\upsilon)$ and the NSNS 2-form $B_2(X,\vartheta,\upsilon)$ as a power series in υ^i. This may be determined in the same way as the Θ-expansion derived in [154] (see for example [155]). Then

$$\mathcal{E}^A \;=\; E^A(X,\vartheta) - \mathrm{i}E\Gamma^A\upsilon - \tfrac{\mathrm{i}}{2}\mathcal{D}\upsilon\Gamma^A\upsilon, \tag{5.7}$$

up to quadratic order in υ. For the NSNS 2-form we have

$$B_2 \;=\; B_2^{\mathrm{coset}}(x,y,\vartheta) - \mathrm{i}E^A \wedge E\Gamma_A\sigma^3\upsilon - \tfrac{1}{2}E\Gamma^A\upsilon \wedge E\Gamma_A\sigma^3\upsilon - \tfrac{\mathrm{i}}{2}E^A \wedge \mathcal{D}\upsilon\Gamma_A\sigma^3\upsilon, \tag{5.8}$$

where $E^{\alpha i}(x,y,\vartheta) = \frac{1}{2}(1-\Gamma^{2389})^\alpha{}_\beta E^{\beta i}(x,y,\vartheta)$, $E^a(x,y,\vartheta)$, and $E^{\hat{a}}(x,y,\vartheta)$ are the supervielbeins constructed in terms of the $\frac{PSU(1,1|2)\times PSU(1,1|2)}{SO(1,2)\times SU(2)}$ supercoset currents while $E^{a'} = \mathrm{d}\varphi^{a'}$ is the flat vielbein along T^4. The Pauli matrix σ^3_{ij} contracts the indices $i,j = 1,2$ of the spinors. Moreover, the NSNS gauge potential B_2^{coset} has again the form as in (4.37), and for the background under consideration the covariant derivative $\mathcal{D}$ is given by

$$\mathcal{D}\upsilon \;=\; \nabla\upsilon - \tfrac{\mathrm{i}}{16\cdot 5!}E^A F_{B_1\cdots B_5}\Gamma^{B_1\cdots B_5}\Gamma_A\sigma^2\upsilon \;=\; \nabla\upsilon - \tfrac{\mathrm{i}}{4}E^A(1-\Gamma^{2389})\Gamma^{01234}\Gamma_A\sigma^2\upsilon,$$

where $\nabla := \mathrm{d} - \frac{1}{4}\Omega^{AB}\Gamma_{AB}$ and $\Omega^{AB}(x,y,\vartheta)$ is the spin connection on $\frac{PSU(1,1|2)\times PSU(1,1|2)}{SO(1,2)\times SU(2)}$ defined in terms of the currents as in section 4.4. Substituting the expressions (5.7) and (5.8) into the string action

(4.6) we arrive at

$$\begin{aligned}\mathcal{L} \;=\; & \mathcal{L}_{\text{coset}} + \tfrac{1}{2}{*}\mathrm{d}\varphi^{a'} \wedge \mathrm{d}\varphi^{a'} - \mathrm{i}{*}\mathrm{d}\varphi^{a'} \wedge E\Gamma_{a'}\upsilon - \mathrm{i}\mathrm{d}\varphi^{a'} \wedge E\Gamma_{a'}\sigma^3\upsilon - \tfrac{1}{2}{*}E\Gamma^{a'}\upsilon \wedge E\Gamma_{a'}\upsilon - \\ & - \tfrac{1}{2}E\Gamma^{a'}\upsilon \wedge E\Gamma_{a'}\sigma^3\upsilon - \tfrac{\mathrm{i}}{2}{*}E^A \wedge \mathcal{D}\upsilon\Gamma_A\upsilon - \tfrac{\mathrm{i}}{2}E^A \wedge \mathcal{D}\upsilon\Gamma_A\sigma^3\upsilon,\end{aligned} \tag{5.9}$$

where we have used the projector properties (5.3). Here, $\mathcal{L}_{\text{coset}}$ has the same form as (2.1) in [36] (with $\gamma = \Gamma^4$). The Lagrangian above contains a lot of υ-dependent terms in the gauge in which the υ are non-zero. These υ contribute to the T-dualisation along the supercoset directions (x^m, θ).

5.3 Self-duality of $AdS_d \times S^d \times S^d \times T^{10-3d}$ Superstrings

In this section we finally address the self-duality of the non-exceptional backgrounds $AdS_d \times S^d \times S^d \times T^{10-3d}$ ($d = 2, 3$) by extending the discussion in section 4.4 as well as the previous section. These backgrounds preserve 1/4 and 1/2 of the 10-dimensional supersymmetry and can be supported by either NSNS or RR fluxes [130, 131, 156–158]. In what follows we shall consider these backgrounds to be supported by the following 3-flux

$$F_3 \;=\; \tfrac{1}{3}(\varepsilon_{cba}e^a \wedge e^b \wedge e^c + \frac{R_{AdS}}{R_+}\varepsilon_{\hat{c}\hat{b}\hat{a}}e^{\hat{a}} \wedge e^{\hat{b}} \wedge e^{\hat{c}} + \frac{R_{AdS}}{R_-}\varepsilon_{c'b'a'}e^{a'} \wedge e^{b'} \wedge e^{c'}), \tag{5.10}$$

where $\hat{a}$ and a' are, respectively the tangent space indices of the two three-spheres and $R_\pm$ are their radii. T-dualising the latter background along the S^1 (a bosonic direction), one gets the type IIA $AdS_3 \times S^3 \times S^3 \times S^1$ with the 4-flux

$$F_4 \;=\; \mathrm{d}\varphi^9 \wedge (\tfrac{1}{3}(\varepsilon_{cba}e^a \wedge e^b \wedge e^c + \frac{R_{AdS}}{R_+}\varepsilon_{\hat{c}\hat{b}\hat{a}}e^{\hat{a}} \wedge e^{\hat{b}} \wedge e^{\hat{c}} + \frac{R_{AdS}}{R_-}\varepsilon_{c'b'a'}e^{a'} \wedge e^{b'} \wedge e^{c'})). \tag{5.11}$$

We will show that the $AdS_d \times S^d \times S^d$ sectors described by supercoset σ-models are T-self-dual under combined bosonic and fermionic T-dualities, provided that T-dualisation involves one of the spheres S^d. The isometries are governed by the exceptional Lie supergroups $D(2,1;\alpha)$ (for $d = 2$) and $D(2,1;\alpha) \times D(2,1;\alpha)$ (for $d = 3$). In general, the corresponding calculations are complex and challenging, as a result we will not present details here. To make computations simpler, we set the non-supersymmetric fermionic modes of the string to zero ($\upsilon = 0$). For the $d = 3$ case we use κ-symmetry to do this, whilst setting them to zero by hand for the $d = 2$ case. Then the T^{10-3d} sector decouples (modulo the Virasoro constraints) leaving the $AdS_d \times S^d \times S^d$ sector free to be concentrated on.

5.3.1 The Self-duality of $AdS_2 \times S^2 \times S^2$

Supercoset Structure

The σ-model on $AdS_2 \times S^2 \times S^2$ is based on the supercoset

$$\frac{D(2,1;\alpha)}{SO(1,1)\times SO(2)\times SO(2)}. \tag{5.12}$$

Before constructing the action to analyse its T-duality properties, let us discuss the Lie superalgebra $\mathfrak{d}(2,1;\alpha)$ of $D(2,1;\alpha)$ first. For general properties of the exceptional Lie superalgebra $\mathfrak{d}(2,1;\alpha)$ see for example [148, 159]. For the 10-dimensional supergravity solutions under consideration, the values of the parameter α are restricted to the interval $[0,1]$. They determine the relation between the radii of $AdS_2\times S^2_+\times S^2_-$,

$$\alpha = \frac{R^2_{AdS}}{R^2_-} \text{ and } 1-\alpha = \frac{R^2_{AdS}}{R_+}. \tag{5.13}$$

To avoid confusion regarding the parameter α and the spinor index α, in what follows we will set $\alpha := \cos^2(\tau) := c^2$ and $1-\alpha := \sin^2(\tau) := s^2$, respectively.

Lie Superalgebra $\mathfrak{d}(2,1;c^2)$

The maximal Grassmann-even subalgebra of the Lie superalgebra $\mathfrak{d}(2,1;c^2)$ is $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(2)$, and we set $\mathfrak{sl}(2,\mathbb{R}) := \langle P,K,D\rangle$, $\mathfrak{su}(2) := \langle L_a\rangle$, and $\mathfrak{su}(2) := \langle R^\alpha{}_\beta\rangle$, respectively, for $a,b,\ldots = 1,2,3$ and $\alpha,\beta,\ldots = 1,2$. The corresponding commutation relations are

$$\begin{aligned} &= P\ ,\quad [D,K] = -K\ ,\quad [P,K] = 2D\ ,\\ [L_+,L_-] &= -2\mathrm{i}L_3\ ,\quad [L_3,L_\pm] = \pm\mathrm{i}L_\pm\ ,\quad L_\pm := \mathrm{i}L_1\pm L_2\ ,\\ [R^\alpha{}_\beta,R^\gamma{}_\delta] &= \mathrm{i}(\delta^\gamma{}_\beta R^\alpha{}_\delta - \delta^\alpha{}_\delta R^\gamma{}_\beta)\ . \end{aligned} \tag{5.14}$$

Furthermore, $\mathfrak{d}(2,1;c^2)$ contains eight fermionic generators which we denote by Q_α, $\hat{Q}_\alpha$, S_α, and $\hat{S}_\alpha$, respectively. Letting $\sigma^{1,2,3}_{\alpha\beta}$ be the Pauli matrices[2], the remaining non-vanishing (anti-)commutation relations of $\mathfrak{d}(2,1;c^2)$ are given by

$$\begin{aligned} \{Q_\alpha,\hat{Q}_\beta\} &= -\sigma^2_{\alpha\beta}P\ ,\quad \{S_\alpha,\hat{S}_\beta\} = -\sigma^2_{\alpha\beta}K\ ,\\ \{Q_\alpha,\hat{S}_\beta\} &= -c^2\sigma^2_{\alpha\beta}L_+\ ,\quad \{\hat{Q}_\alpha,S_\beta\} = c^2\sigma^2_{\alpha\beta}L_-\ ,\\ \{Q_\alpha,S_\beta\} &= -\sigma^2_{\alpha\beta}(D+\mathrm{i}c^2L_3) - \mathrm{i}s^2\sigma^2_{\alpha\gamma}R^\gamma{}_\beta\ ,\\ \{\hat{Q}_\alpha,\hat{S}_\beta\} &= \sigma^2_{\alpha\beta}(D-\mathrm{i}c^2L_3) + \mathrm{i}s^2\sigma^2_{\alpha\gamma}R^\gamma{}_\beta\ ,\\ [P,S_\alpha] &= -\hat{Q}_\alpha\ ,\quad [P,\hat{S}_\alpha] = -Q_\alpha\ ,\quad [K,Q_\alpha] = -\hat{S}_\alpha\ ,\quad [K,\hat{Q}_\alpha] = -S_\alpha\ ,\\ [D,Q_\alpha] &= \tfrac{1}{2}Q_\alpha\ ,\quad [D,\hat{Q}_\alpha] = \tfrac{1}{2}\hat{Q}_\alpha\ ,\quad [D,S_\alpha] = -\tfrac{1}{2}S_\alpha\ ,\quad [D,\hat{S}_\alpha] = -\tfrac{1}{2}\hat{S}_\alpha\ ,\\ [L_3,Q_\alpha] &= \tfrac{\mathrm{i}}{2}Q_\alpha\ ,\quad [L_3,\hat{Q}_\alpha] = -\tfrac{\mathrm{i}}{2}\hat{Q}_\alpha\ ,\quad [L_3,S_\alpha] = -\tfrac{\mathrm{i}}{2}S_\alpha\ ,\quad [L_3,\hat{S}_\alpha] = \tfrac{\mathrm{i}}{2}\hat{S}_\alpha\ ,\\ [L_+,S_\alpha] &= \hat{S}_\alpha\ ,\quad [L_-,\hat{S}_\alpha] = -S_\alpha\ ,\quad [L_-,Q_\alpha] = \hat{Q}_\alpha\ ,\quad [L_+,\hat{Q}_\alpha] = -Q_\alpha\ ,\\ [R^\alpha{}_\beta,T_\gamma] &= -\mathrm{i}(\delta^\alpha{}_\gamma T_\beta - \tfrac{1}{2}\delta^\alpha{}_\beta T_\gamma)\ ,\quad \text{for}\quad T_\alpha \in \{Q_\alpha,\hat{Q}_\alpha,S_\alpha,\hat{S}_\alpha\}\ . \end{aligned} \tag{5.15}$$

[2] We lower and raise Greek indices using $\epsilon_{\alpha\beta} = \mathrm{i}\sigma^2_{\alpha\beta}$ and $\epsilon^{\alpha\beta} = \mathrm{i}\sigma^{2\,\alpha\beta}$ with $\epsilon^{\alpha\gamma}\epsilon_{\gamma\beta} = \delta^\alpha{}_\beta$, so that for example $\sigma^{1\,\alpha}{}_\beta := \mathrm{i}\sigma^{2\,\alpha\gamma}\sigma^1_{\gamma\beta} = -\mathrm{i}\sigma^{1\,\alpha\gamma}\sigma^2_{\gamma\beta}$.

The bosonic generators P, K, D, and L_a are skew-Hermitian while $(R^1{}_1)^\dagger = R^2{}_2$ and $(R^1{}_2)^\dagger = -R^2{}_1$. The following reality conditions hold for the fermionic generators: $Q_1^\dagger = \hat{Q}_2$, $Q_2^\dagger = -\hat{Q}_1$ and $S_1^\dagger = -\hat{S}_2$, $S_2^\dagger = \hat{S}_1$. The superalgebra (5.14) and (5.15) are invariant under these reality conditions. When $c^2 \to 0$ we recover the superalgebra $\mathfrak{psu}(1,1|2)$ associated with $AdS_2 \times S^2 \times T^6$ since in that limit the generators $L_\pm$ and L_3 decouple. This is to be expected since in the limit $\alpha \to 0$ (which is $c^2 \to 0$) we recover the non-exceptional background from its exceptional counterpart. Additionally, non-vanishing components of the invariant form of $\mathfrak{d}(2,1;c^2)$ which are compatible with the above choice of basis are

$$\begin{aligned}
\mathrm{Str}(PK) &= 2\ , \quad \mathrm{Str}(DD) \ = \ 1\ ,\\
\mathrm{Str}(L_+L_-) &= -\tfrac{2}{c^2}\ , \quad \mathrm{Str}(L_3L_3) \ = \ \tfrac{1}{c^2}\ ,\\
\mathrm{Str}(R^\alpha{}_\beta R^\gamma{}_\delta) &= \tfrac{2}{s^2}\left(\delta^\alpha{}_\delta\delta^\gamma{}_\beta - \tfrac12\delta^\alpha{}_\beta\delta^\gamma{}_\delta\right)\ ,\\
\mathrm{Str}(Q_\alpha S_\beta) &= -2\sigma^2_{\alpha\beta}\ , \quad \mathrm{Str}(\hat{Q}_\alpha\hat{S}_\beta) \ = \ 2\sigma^2_{\alpha\beta}\ .
\end{aligned} \tag{5.16}$$

$\mathbb{Z}_4$-grading and order-4 Automorphism

To formulate the supercoset action based on (5.12), we need to fix a $\mathbb{Z}_4$-grading of the superalgebra $\mathfrak{d}(2,1;c^2)\otimes\mathbb{C} \cong \bigoplus_{m=0}^3 \mathfrak{g}_{(m)}$. Our decomposition is as follows

$$\begin{aligned}
\mathfrak{g}_{(0)} &:= \left\langle P+K, L_++L_-, \sigma^1_{\gamma[\alpha}R^\gamma{}_{\beta]}\right\rangle\ ,\\
\mathfrak{g}_{(1)} &:= \left\langle Q_\alpha - \sigma^{1\,\beta}{}_\alpha S_\beta, \hat{Q}_\alpha - \sigma^{1\,\beta}{}_\alpha \hat{S}_\beta\right\rangle\ ,\\
\mathfrak{g}_{(2)} &:= \left\langle P-K, D, L_+-L_-, L_3, \sigma^1_{\gamma(\alpha}R^\gamma{}_{\beta)}\right\rangle\ ,\\
\mathfrak{g}_{(3)} &:= \left\langle Q_\alpha + \sigma^{1\,\beta}{}_\alpha S_\beta, \hat{Q}_\alpha + \sigma^{1\,\beta}{}_\alpha \hat{S}_\beta\right\rangle\ ,
\end{aligned} \tag{5.17}$$

where brackets (respectively, parentheses) indicate normalised anti-symmetrisation (respectively, symmetrisation) of the enclosed indices. Note that $\mathfrak{g}_{(0)} \cong \mathfrak{so}(1,1)\oplus\mathfrak{so}(2)\oplus\mathfrak{so}(2)$ as required. The order-4 automorphism $\Omega : \mathfrak{d}(2,1;c^2) \to \mathfrak{d}(2,1;c^2)$ associated with this $\mathbb{Z}_4$-grading is given by

$$\begin{aligned}
\Omega(P) &= K\ , \quad \Omega(K) \ = \ P\ , \quad \Omega(D) \ = \ -D\ ,\\
\Omega(L_3) &= -L_3\ , \quad \Omega(L_\pm) \ = \ L_\mp\ , \quad \Omega(R^\alpha{}_\beta) \ = \ \sigma^{1\,\alpha\gamma}\sigma^1_{\beta\delta}R^\delta{}_\gamma\ ,\\
\Omega(Q_\alpha) &= -\mathrm{i}\sigma^{1\,\beta}{}_\alpha S_\beta\ , \quad \Omega(\hat{Q}_\alpha) \ = \ -\mathrm{i}\sigma^{1\,\beta}{}_\alpha\hat{S}_\beta\ ,\\
\Omega(S_\alpha) &= -\mathrm{i}\sigma^{1\,\beta}{}_\alpha Q_\beta\ , \quad \Omega(\hat{S}_\alpha) \ = \ -\mathrm{i}\sigma^{1\,\beta}{}_\alpha\hat{Q}_\beta.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathrm{Str}\big[(P\pm K)(P\pm K)\big] &= \pm4\ , \quad \mathrm{Str}(DD) \ = \ 1\ ,\\
\mathrm{Str}\big[(L_+\pm L_-)(L_+\pm L_-)\big] &= \mp\tfrac{4}{c^2}\ , \quad \mathrm{Str}(L_3L_3) \ = \ \tfrac{1}{c^2}\ ,\\
\mathrm{Str}\big[(\sigma^1_{\mu[\alpha}R^\mu{}_{\beta]})(\sigma^1_{\nu[\gamma}R^\nu{}_{\delta]})\big] &= -\tfrac{1}{s^2}\sigma^2_{\alpha\beta}\sigma^2_{\gamma\delta}\ ,\\
\mathrm{Str}\big[(\sigma^1_{\mu(\alpha}R^\mu{}_{\beta)})(\sigma^1_{\nu(\gamma}R^\nu{}_{\delta)})\big] &= -\tfrac{1}{s^2}(\sigma^1_{\alpha\beta}\sigma^1_{\gamma\delta} - \sigma^1_{\alpha\gamma}\sigma^1_{\beta\delta} - \sigma^1_{\alpha\delta}\sigma^1_{\gamma\beta})\ ,\\
\mathrm{Str}\big[(Q_\alpha \pm \sigma^{1\,\gamma}{}_\alpha S_\gamma)(Q_\beta \mp \sigma^{1\,\delta}{}_\beta S_\delta)\big] &= \mp4\mathrm{i}\sigma^1_{\alpha\beta}\ ,\\
\mathrm{Str}\big[(\hat{Q}_\alpha \pm \sigma^{1\,\gamma}{}_\alpha \hat{S}_\gamma)(\hat{Q}_\beta \mp \sigma^{1\,\delta}{}_\beta \hat{S}_\delta)\big] &= \pm4\mathrm{i}\sigma^1_{\alpha\beta}
\end{aligned} \tag{5.18}$$

which follow from (5.16).

Coset Representative and Associated Current

As always, the next step involves choosing a coset representative for the supercoset space (5.12) based on the superalgebra and $\mathbb{Z}$-grading. From (5.14) we can see that the generators P, Q_α, and L_+ are in involution and therefore form an Abelian subalgebra[3]. Consequently, the associated directions are those along which we will perform T-dualisation. Following our general discussion in section 4.3, an appropriate form of the coset representative is[4]

$$\begin{aligned} g &:= \mathrm{e}^{xP+\theta^\alpha Q_\alpha+\lambda_+ L_+}\,\mathrm{e}^B\,\mathrm{e}^{\xi^\alpha S_\alpha}\ , \\ \mathrm{e}^B &:= \mathrm{e}^{\hat\theta^\alpha \hat Q_\alpha+\hat\xi^\alpha \hat S_\alpha}\,|y|^D\,\mathrm{e}^{-\lambda_3 L_3}\,\mathrm{e}^{-\rho^\beta{}_\alpha R^\alpha{}_\beta}\ . \end{aligned} \tag{5.19}$$

where we assume that both λ_+ and λ_3 are complex[5]. We are therefore essentially dealing with the complexification $SL(2,\mathbb{C})/\mathbb{C}^*$ of the coset $SO(3)/SO(2) \cong SU(2)/U(1) \cong S^2$. From the perspective of fermionic T-duality, such a complexification is rather natural (see [22] for a similar case in $AdS_5 \times S^5$). Note that the resulting line element (as seen in Chapter 3) on $SL(2,\mathbb{C})/\mathbb{C}^*$ is

$$ds^2 = \tfrac{1}{4c^2}\big[(\mathrm{d}\lambda_3)^2 + \mathrm{e}^{2\mathrm{i}\lambda_3}(\mathrm{d}\lambda_+)^2\big]. \tag{5.20}$$

Upon performing the change of coordinates $(\lambda_+,\lambda_3) \mapsto (\varphi,\vartheta)$,

$$\begin{aligned} \lambda_+ &= \frac{2\tan(\frac{\vartheta}{2})\sin(\varphi)}{1+2\mathrm{i}\tan(\frac{\vartheta}{2})\cos(\varphi)-\tan^2(\frac{\vartheta}{2})}\ , \\ \mathrm{e}^{-\mathrm{i}\lambda_3} &= \frac{1+\tan^2(\frac{\vartheta}{2})}{1+2\mathrm{i}\tan(\frac{\vartheta}{2})\cos(\varphi)-\tan^2(\frac{\vartheta}{2})} \end{aligned} \tag{5.21}$$

for $\varphi,\vartheta \in \mathbb{C}$, we find the line element

$$(\mathrm{d}s)^2 = \tfrac{1}{4c^2}\big[(\mathrm{d}\vartheta)^2 + \sin^2(\vartheta)\,(\mathrm{d}\varphi)^2\big]\ , \tag{5.22}$$

which, when considering the real slice $\varphi^* = \varphi$ and $\vartheta^* = \vartheta$, becomes the standard line element on the two-sphere S^2. The Maurer–Cartan form $J = g^{-1}\mathrm{d}g$ corresponding to the coset representative (5.19) is of the form

[3] The maximal Abelian subalgebra for $\mathfrak{d}(2,1;c^2)$ has two bosonic and two fermionic generators.

[4] To arrive at the coset representative for $AdS_2 \times S^2 \times T^2$ from the representative (5.19) in the limit $c \to 0$, one first needs to re-scale the coordinates $\lambda_+ \to c\lambda_+$, $\lambda_3 \to c\lambda_3$, and $\rho^\alpha{}_\beta \to s\rho^\alpha{}_\beta$ first. Then, we perform $c \to 0$. Upon taking this limit, the second sphere S^2, whose metric becomes flat, decouples from the $AdS_2 \times S^2$ supercoset and re-compactifies into T^2 which is part of the larger T^6.

[5] This technical assumption is used to help ease the T-duality transformations considered herein.

$$\begin{aligned} J &= \mathrm{e}^{-\xi^\alpha S_\alpha} J^{(0)} \mathrm{e}^{\xi^\alpha S_\alpha} + \mathrm{d}\xi^\alpha S_\alpha \\ &= J^{(0)} - \xi^\alpha \big[S_\alpha, J^{(0)}\big] + \tfrac{\mathrm{i}}{4}\xi^2 \sigma^{2\,\alpha\beta}\big\{S_\alpha, \big[S_\beta, J^{(0)}\big]\big\} + \mathrm{d}\xi^\alpha S_\alpha \ , \end{aligned} \tag{5.23}$$

where, as before, $J^{(0)}$ does not depend on the fermionic coordinate ξ^α, and we have set $\xi^2 := \mathrm{i}\sigma^2_{\alpha\beta}\xi^\alpha\xi^\beta$. The explicit form of the components of the current J are given in Appendix B. Using the $\mathbb{Z}_4$-grading (5.17), the coset current J decomposes according to $J = J_{(0)} + J_{(1)} + J_{(2)} + J_{(3)}$ with

$$\begin{aligned} J_{(0)} &= \tfrac{1}{2}(J_P + J_K)(P + K) + \tfrac{1}{2}(J_{L_+} + J_{L_-})(L_+ + L_-) - J_{R^\alpha{}_\beta}\sigma^{1\,\alpha\gamma}\sigma^1_{\delta[\gamma}R^\delta{}_{\beta]} \ , \\ J_{(1)} &= \tfrac{1}{2}(J_{Q_\alpha} - \sigma^{1\,\alpha}{}_\beta J_{S_\beta})(Q_\alpha - \sigma^{1\,\beta}{}_\alpha S_\beta)\tfrac{1}{2}(J_{\hat{Q}_\alpha} - \sigma^{1\,\alpha}{}_\beta J_{\hat{S}_\beta})(\hat{Q}_\alpha - \sigma^{1\,\beta}{}_\alpha \hat{S}_\beta) \ , \\ J_{(2)} &= \tfrac{1}{2}(J_P - J_K)(P - K) + J_D D + \tfrac{1}{2}(J_{L_+} - J_{L_-})(L_+ - L_-) + J_{L_3}L_3 - \\ &\quad - J_{R^\alpha{}_\beta}\sigma^{1\,\alpha\gamma}\sigma^1_{\delta(\gamma}R^\delta{}_{\beta)} \ , \\ J_{(3)} &= \tfrac{1}{2}(J_{Q_\alpha} + \sigma^{1\,\alpha}{}_\beta J_{S_\beta})(Q_\alpha + \sigma^{1\,\beta}{}_\alpha S_\beta) + \tfrac{1}{2}(J_{\hat{Q}_\alpha} + \sigma^{1\,\alpha}{}_\beta J_{\hat{S}_\beta})(\hat{Q}_\alpha + \sigma^{1\,\beta}{}_\alpha \hat{S}_\beta) \ . \end{aligned} \tag{5.24}$$

Supercoset Action

Using the $\mathbb{Z}_4$-grading (5.17) together with the currents (5.24) and the invariant form (5.16), the σ-model action becomes

$$\begin{aligned} S = -\tfrac{T}{2}\int_\Sigma \Big\{ &- *(J_P - J_K)\wedge(J_P - J_K) + *J_D \wedge J_D + \\ &+ \tfrac{1}{c^2} * (J_{L_+} - J_{L_-})\wedge(J_{L_+} - J_{L_-}) + \tfrac{1}{c^2} *J_{L_3}\wedge J_{L_3} + \\ &+ \tfrac{1}{s^2}(*J_{R^\alpha{}_\beta}\wedge J_{R^\beta{}_\alpha} - \sigma^{1\,\alpha\gamma}\sigma^1_{\beta\delta} *J_{R^\alpha{}_\beta}\wedge J_{R^\gamma{}_\delta}) - \\ &- \mathrm{i}\sigma^1_{\alpha\beta}\big(J_{Q_\alpha}\wedge J_{Q_\beta} + J_{S_\alpha}\wedge J_{S_\beta} - J_{\hat{Q}_\alpha}\wedge J_{\hat{Q}_\beta} - J_{\hat{S}_\alpha}\wedge J_{\hat{S}_\beta}\big)\Big\} \ . \end{aligned} \tag{5.25}$$

For the 'non-exceptional' cases, the matrix $(-\sigma^1)$ can be identified with the matrix $\Gamma^4\mathbb{P}$ along the AdS_2 radial direction with $\mathbb{P}$ being the projector which singles out eight unbroken supersymmetries of the background under consideration. The same is true for the exceptional cases.

T-dualisation

We are now ready to perform the T-duality on the action (5.25) according to the procedure laid out in section 4.3. After some technical algebra, a field re-definition and the use of the Maurer-Cartan equations we find that the resulting dual action has the same form as the initial action, but with currents constructed with the different coset element

$$\tilde{g} := \mathrm{e}^{\tilde{x}K - \mathrm{i}\sigma^{2\,\alpha\beta}\tilde{\theta}_\alpha S_\beta + \tilde{\lambda}_+ L_-}\,\mathrm{e}^B\,\mathrm{e}^{\sigma^{1\beta}{}_\alpha\xi^\alpha Q_\beta}, \tag{5.26}$$

where, e^B is the same as in (5.19). Therefore, the supercoset sigma model on $AdS_2 \times S^2 \times S^2$ is self-dual under the combined T-dualities along x, θ^α, and λ_+. As a check, in the limit $c^2 \to 0$, upon an appropriate re-scaling of the J_L-currents, the action reduces to the $\frac{PSU(1,1|2)}{SO(1,1)\times U(1)}$ supercoset σ-model for $AdS_2 \times S^2$. In this limit, the dualised sphere S^2 gets 'decompactified' into a T^2 torus which completely decouples from the $AdS_2 \times S^2$ and fermionic sector.

5.3.2 Self-duality for $AdS_3 \times S^3 \times S^3$

Considering the subsector of the $AdS_3 \times S^3 \times S^3 \times S^1$ theory in which the string moves only in $AdS_3 \times S^3 \times S^3$ while its non-supersymmetric fermionic modes are gauge fixed to zero and the S^1-fluctuations decouple from the rest. For this case, the T-dualisation process is almost identical to the discussion above for the $AdS_2 \times S^2 \times S^2$ case, however, the explicit calculations are technically more involved. Therefore, we only outline the basic steps here.

The supercoset sigma model on $AdS_3 \times S^3 \times S^3$ is based on the supercoset

$$\frac{\mathrm{D}(2,1;c^2)\times \mathrm{D}(2,1;c^2)}{SO(1,2)\times SO(3)\times SO(3)} \;. \tag{5.27}$$

The Lie superalgebra $\mathfrak{d}(2,1;c^2)\oplus\mathfrak{d}(2,1;c^2)$ has $\{P_m, D, K_m, L_a^\pm, R^{\pm\, i}{}_j\}$ for $m = 0,1$, $a = 1,2,3$, and $i,j = 1,2$ as its bosonic generators and $\{Q_{i\alpha}, S_{i\alpha}, \hat{Q}_{i\alpha}, \hat{S}_{i\alpha}\}$ for $\alpha = 1,2$ as its fermionic generators, respectively. Here, the $L_a^\pm$ and $R^{\pm\, i}{}_j$ are the generators of $\mathfrak{so}(3)\oplus\mathfrak{so}(3)\oplus\mathfrak{so}(3)\oplus\mathfrak{so}(3)$. Furthermore, the generators $\{P_m, Q_{i\alpha}, L^\pm := \mathrm{i}L_1^\pm + L_2^\pm\}$ are in involution[6] so that the coset representative will have the left factor of the form $\mathrm{e}^{x^m P_m + \theta^{i\alpha}Q_{i\alpha} + \lambda_+ L^+ + \lambda_- L^-}$. The coordinates x^m parametrize the 2-dimensional Minkowski boundary of AdS_3. As for the $AdS_2\times S^2\times S^2$ case, we shall work with the complexification $SO(4,\mathbb{C})/SO(3,\mathbb{C})$ of $SO(4)/SO(3) \cong [SU(2)\times SU(2)]/SU(2) \cong SU(2) \cong S^3$. Thus, the coordinates $\lambda_\pm$ are assumed to be complex. The resulting line element on $SO(4,\mathbb{C})/SO(3,\mathbb{C})$ will be of the form

$$ds^2 \;=\; \tfrac{1}{4c^2}\left[(\mathrm{d}\lambda_3)^2 + \mathrm{e}^{2\mathrm{i}\lambda_3}(\mathrm{d}\lambda_+)^2 + \mathrm{e}^{2\mathrm{i}\lambda_3}(\mathrm{d}\lambda_-)^2\right] \;. \tag{5.28}$$

The next step is to choose an appropriate $\mathbb{Z}_4$-grading for the space (5.27). The T-duality is then performed along the bosonic directions x^m and $\lambda_\pm$ and the fermionic directions $\theta^{i\alpha}$. The T-self-duality of the supercoset σ-model follows. We have explicitly checked this up to the second order in the four-component fermions $\xi^{i\alpha}$, like in the $\mathrm{AdS}_2\times S^2\times S^2$ case. We believe that the invariance holds to the highest (4th-order) in $\xi^{i\alpha}$. This is supported by the fact that at $\alpha = 0$, the model reduces to the $\mathrm{AdS}_3\times S^3$ supercoset sigma model times the torus sector, which have shown to be duality invariant.

[6]Note that the maximal Abelian subalgebra of $\mathfrak{d}(2,1;c^2)\oplus\mathfrak{d}(2,1;c^2)$ has four bosonic and four fermionic generators.

5.4 Combined bosonic-fermionic T-duality of the Ramond-Ramond $AdS_d \times S^d \times M^{10-2d}$ backgrounds

This section provides additional evidence for the T-self-duality of the complete $AdS_3 \times S^3 \times S^3 \times S^1$ theory by proving invariance under the combined bosonic and fermionic T-duality of its supergravity background. We shall apply the T-duality rules directly to the corresponding supergravity component fields. We are therefore extending the earlier results of [16, 22, 23, 101, 107] to the whole class of the Ramond-Ramond $AdS_d \times S^d \times M^{10-2d}$ super-backgrounds. This general approach deals with all the backgrounds considered in this thesis at once.

5.4.1 Rules for Fermionic T-duality

Killing Spinors

Following the procedure laid out in Chapter 1 which was based on the conventional rules [12,13, 129] (and [66] for a generalization to the whole superspace), we may T-dualise the bosonic directions. After this procedure was generalized for fermionic directions in [22], we found out that we could T-dualise along Grassman-even Killing spinors, denoted by $\Xi_\mu(X)$, which generate Abelian super-isometries. Here μ labels the number of the Killing spinor. Recall that fermionic T-duality acts on the dilaton $\Phi(X)$ and the RR fields, but leaves the metric and the NSNS 2-form invariant. That we dualise along Grassman-even directions implies the following condition

$$\Xi_\mu \Gamma_A \Xi_\nu \;=\; 0 \quad \text{for all} \quad A, \mu, \nu \quad \text{with} \quad A = 0, 1, \ldots, 9. \tag{5.29}$$

This condition has non-trivial solutions if the Killing spinors are complex, thus manifesting the fact that they are associated with complex Grassmann-odd directions in superspace. The Killing spinor conditions themselves have the following form

$$\begin{aligned} \partial_M \Xi - \tfrac{1}{4}\Omega_M^{AB}(X)\Gamma_{AB}\Xi \;&=\; -\tfrac{1}{8}\not{F}\mathcal{E}_M^A(X)\Gamma_A\Xi\,, \\ \tfrac{1}{16}\Gamma_A \not{F}\Gamma^A\Xi \;&=\; 0 \end{aligned} \tag{5.30}$$

where $\Omega_M^{AB}(X)$ and $\mathcal{E}_M^A(X)$ are the spin connection and the bosonic vielbeins of the 10-dimensional background respectively, and $\not{F}$ denotes the contribution of the RR fluxes

$$\not{F} \;=\; \begin{cases} e^{\Phi}\big(\frac{1}{2}F^{(2)}_{AB}\Gamma^{AB}\Gamma_{11} + \frac{1}{4!}F^{(4)}_{ABCD}\Gamma^{ABCD}\big) & \text{type IIA} \\ -\frac{e^{\Phi}}{2}(1+\Gamma^{11})\big(\mathrm{i}F^{(1)}_A\Gamma^A\sigma^2 + \frac{1}{3!}F^{(3)}_{ABC}\Gamma^{ABC}\sigma^1 + \frac{\mathrm{i}}{2\cdot 5!}F^{(5)}_{A\cdots E}\Gamma^{A\cdots E}\sigma^2\big) & \text{type IIB} \end{cases} \tag{5.31}$$

We arrive at (5.30) by requiring that the supersymmetry variations for the dilatino and gravitino vanish. They are also determined by the individual geometry and its accompanying RR fluxes. Requiring that the first equation in (5.30) be integrable determines the projector $\mathcal{P}_{8(d-1)}$. This singles out $8(d-1)$

fermionic isometries for the background considered. The second equation in (5.30) is then identically satisfied.

Fermionic T-duality Rules

Upon solving for the Killing spinor equations (5.30), one can derive from[7]

$$\partial_M \mathcal{C}_{\mu\nu} \;=\; \begin{cases} E^A_M \bar{\Xi}_\mu \Gamma_A \Gamma^{11} \Xi_\nu & \text{type IIA} \\ E^A_M \bar{\Xi}_\mu \Gamma_A \sigma^3 \Xi_\nu & \text{type IIB} \end{cases} \tag{5.32}$$

the matrix $\mathcal{C} = (\mathcal{C}_{\mu\nu}(X))$ which is formed by the components of the NSNS 2-form B_2 along the Abelian fermionic isometries, that is,

$$\mathrm{d}\theta^\mu \wedge \mathrm{d}\theta^\nu B_{\mu\nu}(X,\Theta)|_{\Theta=0} \;:=\; \mathrm{d}\theta^\mu \wedge \mathrm{d}\theta^\nu \mathcal{C}_{\mu\nu}(X). \tag{5.33}$$

Once the matrix $\mathcal{C}_{\mu\nu}$ is known, one obtains a shift of the dilaton under fermionic T-duality

$$\Delta\Phi \;=\; \Phi' - \Phi \;=\; \tfrac{1}{2}\log(\det \mathcal{C}) \tag{5.34}$$

and of the RR fields, which in our conventions is

$$\Delta F \;=\; \not{F}' - \not{F} \;=\; 8\Xi_\mu (\mathcal{C}^{-1})^{\mu\nu} \Xi_\nu \Gamma, \tag{5.35}$$

where Γ is a certain product of Γ-matrices that is used to split the fermionic $E^{(1,2)}$ currents into four pieces corresponding to the splitting up of the superalgebra generators $\mathcal{Q}$ into Q, $\hat{Q}$, S, and $\hat{S}$, respectively. In particular, backgrounds with a 5-flux alone, have $\Gamma = 1$. For $AdS_d \times S^d \times M^{10-2d}$ (with $d = 2, 3$) with 3-flux, we have $\Gamma = -\Gamma^{23}$. For backgrounds with both 2- and 4-fluxes, we have $\Gamma = \Gamma^{11}\Gamma^{123}$, while for backgrounds with 4-flux only, we have $\Gamma = \Gamma^1$.

Explicit Form of the Killing Spinors

A direct way to get the explicit form of the Killing spinors is to read them off from the corresponding components of the fermionic currents J_Q associated with the generators Q of the superisometry algebra. By construction, the Killing spinors satisfy the defining relations (5.29) and (5.30) and are the components of $J_\alpha{}^\beta(|y|, y^{\hat{a}}, \lambda_3)$ in

[7] Equations (5.32) determine the components H_M of the field strength $H_3 = dB_2$ of the NSNS 2-form for the super-backgrounds under consideration.

$$J_{Q_\alpha}|_{\Theta=0} \;=\; \mathrm{d}\theta^\mu J_\mu{}^\alpha(|y|, y, \lambda_3) \;=\; \mathrm{d}\theta^\mu \mathrm{e}^{-B} Q_\mu \mathrm{e}^B|_{Q_\alpha, \hat\theta=\hat\xi=0} \;\stackrel{!}{=}\; \mathrm{d}\theta^\mu \Xi_\mu{}^\alpha, \tag{5.36}$$

where e^B was defined in the last section . The Killing spinor condition, a particular form of (5.30), is obtained by differentiation of (5.36)

$$\begin{aligned} \mathrm{d}\Xi_\mu + \left[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^B, \Xi_\mu\right]|_{\Theta=0} \;&=\; 0 \;, \\ \mathrm{e}^{-B}\mathrm{d}\mathrm{e}^B|_{\Theta=0} \;&=\; \Omega^{\hat a\hat b}(y/|y|) R_{\hat a\hat b} + J_D(|y|) D + J_{L_3}(\lambda_3) L_3 \;. \end{aligned} \tag{5.37}$$

Note that the index μ should be regarded as an external one, labelling the number of each Killing spinor. Considering the structure of the coset element $\mathrm{e}^{B(|y|,y,\lambda_3)}$ and the commutation relations $[D, Q] = \frac{1}{2}Q$, $[R_{\hat a}, Q] = -\frac{s^2}{2} Q\Gamma_{\hat a}\Gamma^4\mathbb{P}$, and $[L_3, Q] = \frac{\mathrm{i}}{2}Q$, we have the following generic form of the Killing spinors in question[8]

$$\Xi_\mu{}^\alpha \;=\; J_\mu{}^\alpha(|y|, y, \lambda_3) \;=\; |y|^{-\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}}{2}c\lambda_3} \mathcal{O}_\mu{}^\alpha(y^{\hat a}/|y|), \tag{5.38}$$

where $\mathcal{O}_\mu{}^\alpha(y^{\hat a}/|y|) := (\mathrm{e}^{s\mathbb{P}\Gamma_{\hat a}\Gamma_4\, y^{\hat a}/(2|y|)})_\mu{}^\alpha$ is a $Spin(d+1)$-matrix associated with the coset $S^d \cong SO(d+1)/SO(d)$ and $\mathbb{P} := \mathbb{P}_+\mathcal{P}_{8(d-1)}$ is the projector matrix which singles out the $2(d-1)$ anti-commuting isometries $Q = Q\mathbb{P}$ for each case of $AdS_d \times S^d \times M^{10-2d}$, as was described in the previous sections. By definition, we have

$$\mathcal{O}^T\Gamma_4\mathcal{O} \;=\; \Gamma_4\mathbb{P}, \tag{5.39}$$

The structure of the matrix $\mathcal{C}_{\mu\nu}$, see (5.33), is read off from the form of the Wess-Zumino term of the Green-Schwarz superstring action, which in our conventions has the form

$$B_{\mu\nu}|_{\Theta=0} \;=\; \mathrm{i}J_\mu{}^\gamma \Gamma^4_{\gamma\delta} J_\nu{}^\delta(|y|, y, \lambda_3) \;\stackrel{!}{=}\; \mathcal{C}_{\mu\nu}. \tag{5.40}$$

Using (5.38) and (5.39), we find that

$$\mathcal{C}_{\mu\nu} \;=\; \mathrm{i}|y|^{-1}\mathrm{e}^{\mathrm{i}c\lambda_3}(\Gamma^4\mathbb{P})_{\mu\nu} \tag{5.41}$$

and its inverse is

$$\mathcal{C}^{-1\mu\nu} \;=\; -\mathrm{i}|y|\mathrm{e}^{-\mathrm{i}c\lambda_3}(\mathbb{P}\Gamma^4)^{\mu\nu}. \tag{5.42}$$

From (5.41) we can read off the shift (5.34) of the dilaton

[8] To have a smooth limit from $AdS_d \times S^d \times S^d \times T^{10-3d}$ to $AdS_d \times S^d \times T^{10-2d}$ at $c \to 0$, we had rescale the coordinates $\lambda_\pm$ and λ_3 of the second sphere.

$$\Delta\Phi \;=\; \tfrac{1}{2}\log(\det\mathcal{C}) \;=\; -(d-1)\log|y| + \mathrm{i}(d-1)c\lambda_3 \tag{5.43}$$

and from (5.42) we read off the change (5.35) in the RR fluxes upon the fermionic T-duality for all the considered cases

$$\Delta F \;=\; 8J_\mu\mathcal{C}^{-1\mu\nu}J_\nu\Gamma \;=\; -8\mathrm{i}\mathbb{P}\Gamma^4\Gamma \;=\; -(1+\mathrm{i}\Gamma^{0123})\,\not{F}\,. \tag{5.44}$$

Explicit Form of the Ramond-Ramond Fluxes

In the interest of completeness, let us present more details on the form of the RR fluxes characterized by (5.44) in some of the exceptional $\mathrm{AdS}_d \times S^d \times S^d \times T^{10-3d}$ cases:

(a) $AdS_3 \times S^3 \times S^3 \times S^1$: Consider type IIB theory with a 3-flux as in [127], for example

$$\not{F}_3 \;=\; 2\left(\Gamma^{014} + \sqrt{\alpha}\Gamma^{823} + \sqrt{1-\alpha}\Gamma^{567}\right) \;=\; 4\mathcal{P}_{16}\Gamma^{014}\,. \tag{5.45}$$

In this case, $\Gamma = -\Gamma^{23}$ as in the corresponding non-exceptional $\alpha = 0$ case. Alternatively, we may T-dualise this background along the S^1-coordinate φ^9 to get the IIA background with 4-flux only, as written in [121], and use the same $\mathcal{P}_{16}$ with $\Gamma = \Gamma^{239}$.

(b) $AdS_2 \times S^2 \times S^2 \times T^4$: Consider type IIA theory with a 4-flux, then we may write the corresponding projector of rank 8 as a product of two rank 16 projectors, $\mathcal{P}_8 = P_1 P_2$, as in [121]. Re-numbering the 4-flux components of [121] such that $0,\ldots,3$ are the directions along which we dualise (with $2,3,8,9$ the T^4 directions, one sphere being parametrised by $x^7 = \lambda_3$ and $x^1 = \lambda_+$, and the other sphere directions labelled by $5,6$) this reads

$$\begin{gathered}\not{F}_4 \;=\; 4P_1P_2\Gamma^{04\,92}\,,\\ P_1 \;:=\; \tfrac{1}{2}(1+\Gamma^{9238})\,, \qquad P_2 \;:=\; \tfrac{1}{2}(1+\sqrt{\alpha}\Gamma^{04\,71\,23}+\sqrt{1-\alpha}\Gamma^{04\,56\,98})\end{gathered} \tag{5.46}$$

with $\Gamma = \Gamma^{239}$.

In all the cases under consideration, the shifts (5.43) and (5.44) are undone by the corresponding bosonic T-dualities, as we shall show next for the $AdS_d \times S^d \times M^{10-2d}$ backgrounds.

5.4.2 Compensating Bosonic T-duality

General Case

The complete Buscher rules for bosonic T-duality are part of the $O(D,D)$ symmetry of generalised geometry [133]. However, in the cases of interest to us here, the antisymmetric NSNS B-field vanishes and the metric is diagonal, simplifying the rules greatly. Let $\mathcal{I}$ be the set of directions along which we dualise, then the new metric has the components

$$G'_{tt} = \frac{1}{G_{tt}}\,, \qquad t \in \mathcal{I}$$

and remains unchanged in all other directions. The shift in the dilaton is given by (minus half of the) log of the determinant of this block, that is

$$\Delta\Phi = -\tfrac{1}{2}\log\det G_{MN} = -\tfrac{1}{2}\sum_{t\in\mathcal{I}}\log G_{tt}. \tag{5.47}$$

Allowing for a some abuse in notation that t refers to flat directions here, we can write the change in the RR forms as

$$\not{F}'' = \left(\prod_{t\in\mathcal{I}} c_t\,\Gamma^t\right)\not{F}'\,, \qquad c_t := \begin{cases} -\mathrm{i} & \text{for } t=0 \\ 1 & \text{else,} \end{cases} \tag{5.48}$$

where $\not{F}'$ is the result of the fermionic T-duality (see (5.35)). The change in the RR fluxes has been written in terms of potentials in [81]. Here, it is important to include factors e^{Φ} as the dilaton is non-trivial. For non-vanishing dilaton and a general metric (i.e. not diagonal), [52] write the changes in the RR fluxes in terms of the field strengths. Time-like T-duality results in imaginary forms [83]. The overall sign is not physical, but depends on the order in which we perform the T-dualities. Let us apply this to the backgrounds of interest to us. T-duality was performed along directions labelled by $t \in 0,...,3$. Then, the T-dualised RR fluxes (5.48) take the form

$$\not{F}'' = -\mathrm{i}\Gamma^{0123}\not{F}'.$$

Substituting into the above equation $\not{F}' = \not{F} + \Delta F$ with the shift ΔF produced by the fermionic T-duality as in (5.44), we see that the combined bosonic-fermionc T-duality leaves the RR fluxes invariant, for example $\not{F}'' = \not{F}$.

The AdS_d Metric

Consider the T-dualisation of the background metric and the dilaton. With our choice of the coset element and corresponding $AdS_d \times S^d$ metric, the effect of applying T-duality along all $d-1$ boundary directions of AdS_d on the line element on AdS_d is

$$\begin{aligned} ds^2 &= \frac{-(\mathrm{d}x^0)^2 + \sum_{i=1}^{d-2}\mathrm{d}x^i\mathrm{d}x^i + \sum_{r=1}^{d+1}\mathrm{d}y^r\mathrm{d}y^r}{|y|^2} \\ &\to |y|^2\Big[-(\mathrm{d}x^0)^2 + \sum_{i=1}^{d-2}\mathrm{d}x^i\mathrm{d}x^i\Big] + \frac{\sum_{r=1}^{d+1}\mathrm{d}y^r\mathrm{d}y^r}{|y|^2} \end{aligned}$$

and the dilaton shift is

$$\Delta_{AdS}\Phi \;=\; (d-1)\log|y|.$$

We can return the metric to its original form by defining $y'^r = y^r/|y|$ which sends $|y| = \sqrt{\sum_{r=1}^{d+1} y^r y^r} \to \frac{1}{|y|}$. Dualising along some torus directions has no effect on the metric or the dilaton.

The S^d Metric

In the exceptional cases $AdS_d \times S^d \times S^d \times T^{10-3d}$ we also dualise along some directions of one of the spheres: λ_+ for $d=2$, and $\lambda_\pm$ for $d=3$. The effect on the line element on S^d, after the usual rescaling,

$$\begin{aligned} ds^2 &= \tfrac{1}{4}\Big[(\mathrm{d}\lambda_3)^2 + \mathrm{e}^{2\mathrm{i}c\lambda_3}(\mathrm{d}\lambda_+)^2 + \underbrace{\mathrm{e}^{2\mathrm{i}c\lambda_3}(\mathrm{d}\lambda_-)^2}_{\text{only for } d=3}\Big] \\ &\to \tfrac{1}{4}\Big[(\mathrm{d}\lambda_3)^2 + \mathrm{e}^{-2\mathrm{i}c\lambda_3}(\mathrm{d}\lambda_+)^2 + \underbrace{\mathrm{e}^{-2\mathrm{i}c\lambda_3}(\mathrm{d}\lambda_-)^2}_{\text{only for } d=3}\Big] \end{aligned}$$

and we recover the original metric by defining $\lambda_3' = -\lambda_3$. The effect on the dilaton is

$$\Delta_S\Phi \;=\; -\mathrm{i}(d-1)c\lambda_3 + (d-1)\log 2. \tag{5.49}$$

Finally, $\Delta_{AdS}\Phi + \Delta_S\Phi$ cancels the fermionic dualitys shift in the dilaton, (5.43) (modulo the constant term which can be ignored). Since the only contribution from the dilaton is as an overall factor multiplying the action in the path integral, it will not affect the classical supergravity argument. In summary, we have shown that the $AdS_d \times S^d \times M^{10-2d}$ backgrounds with $(d = 2, 3, 5)$ are invariant under the combined fermionic-bosonic T-duality.

5.5 Summary

This chapter dealt with the less than maximally supersymmetric backgrounds in the same way that $AdS_5 \times S^5$ was treated at the end of Chapter 4. Our goal was to establish T-self-duality for such backgrounds with and without κ-symmetry gauge fixing. To this end, we were successful. We showed that the background $AdS_3 \times S^3 \times T^4$ was self-dual without fixing κ-symmetry gauge up to second order in ξ, and argued the same was true up to highest (4th order) in ξ. Furthermore, we showed that the exceptional backgrounds $AdS_d \times S^d \times S^d \times M^{10-3d}$ (for $d = 2, 3$) were self-dual in a gauge in which the non-supercoset fermions were fixed to zero, either by hand or using κ-symmetry.

Part IV

Summary and Conclusion

Conclusion

An integrable system is one which is exactly solvable, that is, we can obtain explicit solutions. In the context of string theory and field theory, there are an finite number of conserved charges. Here, integrability implies that there is an associated infinite dimensional algebra. Consider $AdS_5 \times S^5$ which is integrable and dual to $\mathcal{N} = 4$ super Yang-Mills. The presence of integrability makes it reasonable to expect that a complete solution for the spectrum of planar scattering amplitudes of $AdS_5 \times S^5$ may be obtained [21]. Similarly, this is the case for $\mathcal{N} = 4$ super Yang-Mills and its spectrum of dimensions of the gauge invariant generators. Moreover, integrability is expected to be connected with self-duality in the some way. This is not yet well understood [22]. Simply put, integrability places tight constraints on a theory. We would like to see the conserved charges put to use to completely fix all amplitudes. In [26], T-self-duality was used for the AdS_5 case to construct classical solutions for open strings which were related to strong coupling limits of gluon scattering amplitudes [21].

Duality is a desirable property, however, it is difficult to show in general. This is where integrability supplements duality. Integrability, as mentioned already, imposes constraints on the theory. Therefore, more integrability will impose more constraints. These constraints will be seen on both sides of the duality. This is also difficult to show. T-duality has the advantage of being algorithmic. This means that it is much easier to implement. In this thesis we have tried to connect the triangle:

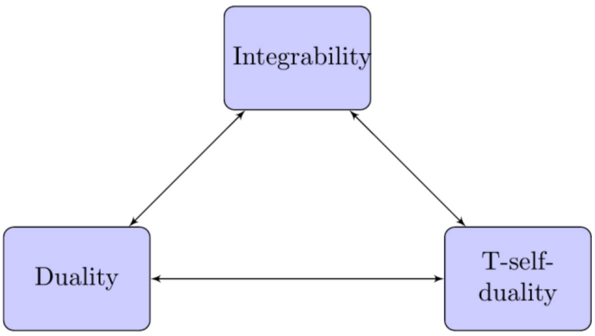

Figure 5.2: Triangle of the relationships between various concepts that we have studied.

The central conjecture made in this thesis may be given by the following statement: T-self-duality $\Longleftrightarrow$ integrability. This 'if and only if' statement was hinted at in [22], but, is this conjecture true for every 10-dimensional AdS-superstring background? Also, in [22], self-duality is proven to all orders in α' for the integrable background $AdS_5 \times S^5$. The proof was elegant, and $AdS_5 \times S^5$ acts as the ideal laboratory. However, our tests work out too well in this laboratory. With tests of generality looming regarding our proposed conjecture, we face the question: Will our conjecture work for backgrounds with less than maximal supersymmetry? We needed to determine whether $AdS_5 \times S^5$ was a unique case. Thus, we need to break supersymmetry, *systematically*. Before moving on, we note that the proof of self-duality in [22] was completed using a partially fixed κ-symmetry gauge. In this thesis, based on the work in [36], we prove that $AdS_5 \times S^5$ is *exactly* self-dual without fixing κ-symmetry at all.

Breaking a little supersymmetry to start, the next to maximally supersymmetric background we arrive at is $AdS_4 \times \mathbb{C}P^3$. As is the case for the maximally supersymmetric $AdS_5 \times S^5$ case, $AdS_4 \times \mathbb{C}P^3$ is integrable in the planar limit [42]. Furthermore, there is a lot of evidence available which favours the presence of dual superconformal symmetry. Thus, it seems likely that the background would be a good candidate for our conjecture. However, the results are disappointing in this regard. $AdS_4 \times \mathbb{C}P^3$ is not self-dual under a combined set of bosonic and fermionic T-duality. We can say that the background is self-dual classically, but at the quantum level we are at a loss. This means that we cannot use the classical self-duality of $AdS_4 \times \mathbb{C}P^3$ to account for the dual superconformal symmetry of ABJM [42]. This brings us to a key question and its associated conjecture: "Why is $AdS_4 \times \mathbb{C}P^3$ not self-dual, despite $AdS_5 \times S^5$ being self-dual?" It is odd that, for some reason, $AdS_4 \times \mathbb{C}P^3$ fails the conjecture despite its many other similarities to $AdS_5 \times S^5$. Now we have the following logic.

The maximal case has the property that it is self-dual and integrable. However, when considering a background with less supersymmetry like $AdS_4 \times \mathbb{C}P^3$, we find that it is not self-dual. We wanted to test whether the absence of self-duality was as a result of $AdS_4 \times \mathbb{C}P^3$ possessing less than maximal supersymmetry. The work covered in Chapters 3-5 addressed this question. We considered two classes of backgrounds: $AdS_d \times S^d \times T^{10-2d}$ $(d = 2, 3)$ and $AdS_d \times S^d \times S^d \times T^{10-3d}$ $(d = 2, 3)$. Both backgrounds have less supersymmetry than $AdS_4 \times \mathbb{C}P^3$ and we proved that they were all self-dual from the supergravity and worldsheet perspectives. Therefore, our claim that $AdS_d \times S^d \times T^{10-2d}$ $(d = 2, 3)$ does not work due to less supersymmetry was incorrect. Some other mechanism, which we do not yet understand, is at play.

Recently, Ó Colgain and Pittelli released a fascinating paper [42] dealing with issues surrounding the self-duality of $AdS_4 \times \mathbb{C}P^3$. Interestingly, they showed that $AdS_4 \times \mathbb{C}P^3$ could not be self-dual using either the supergravity or worldsheet approach. Irrespective of the chosen isometries, one encounters a singularity in the dilaton shift. This results because the generators of the Lie algebra do not admit a non-singular order-4 automorphism [42, 45, 47]. One might then decide to deform the background in such a way that the supersymmetry is preserved, resulting in a new background in the process. This is done via a TsT transformation which involves a Tduality, then a shift (or translation), then a further T-duality. However, as stated in [42], TsT transformations commute with fermionic T-duality. This means that even though we change the background through deformation and without breaking supersymmetry, the conclusion above still holds. This background is not self-dual.

These results preclude the $AdS_4 \times \mathbb{C}P^3$ geometry from being self-dual based on fermionic T-duality. However, a puzzling aspect remains despite this new knowledge. There is still a lot of evidence suggesting that there should be a self-duality transformation at work, based on perturbative observations linked to integrability. Thus, what we learn from this is that the T-duality transformations in [22] are not responsible for the self-duality of supercosets represented by quotient groups or the TsT deformations of their $AdS_5 \times S^5$ case. This, despite $AdS_5 \times S^5$ being integrable. Currently, an explanation eludes us.

Moving forward, there remain a number of outstanding questions. What is responsible for the correct self-duality, i.e. what transformations do we require given the supposition that some sort of self-duality is at play? Do we need to use complexified $\mathcal{C}P^3$ coordinates? What happens when we consider non-trivial

NSNS 2-forms? Evidently, even after nearly three decades of intense scrutiny, the idea of T-duality in string remains as fruitful an area of research as ever. We look forward to exploring these and more ideas further.

Part V

Appendices

Appendix A

Conventions

A.1 Pauli Matrices

The Pauli matrices are defined as

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

with the following algebra

$$\sigma_i \sigma_j = \delta_{ij} \mathbf{1} + 2\epsilon_{ijk}\sigma_k,$$

and the $SU(2)$ generators are given by

$$t^i = \frac{1}{\sqrt{2}}\sigma^i,$$

such that

$$\mathrm{Tr}(t^i t^j) = \delta^{ij}, \quad [t^i, t^j] = i f^{ij}_k t^k = i\sqrt{2}\epsilon^{ij}_k t^k.$$

A.2 Gamma Matrices

We follow the conventions laid out in [16, 82]. Namely, we work with the 32×32 dimensional gamma matrices which represent $\mathbb{R}^{1,9}$ with a mostly plus metric signature, $(- + ... +)$. The Levi-Civita tensor is defined with $\epsilon_{0\ldots9} = 1$. The gamma matrices have the following structure

$$\Gamma^0 = i\sigma_2 \otimes \mathbf{1}_{16} = \begin{bmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{bmatrix}, \quad \Gamma^m = \sigma_1 \otimes \gamma^m = \begin{bmatrix} 0 & (\gamma^m)_{\alpha\beta} \\ (\gamma^m)_{\alpha\beta} & 0 \end{bmatrix}$$

with $(\Gamma^0)^2 = -1$ and $(\Gamma^m)^2 = 1$. The 16×16 γ matrices are as follows

$$\begin{aligned}
\gamma^1 &= \sigma^2 \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma^2 \\
\gamma^2 &= \sigma^2 \otimes \mathbf{1} \otimes \sigma^1 \otimes \sigma^2 \\
\gamma^3 &= \sigma^2 \otimes \mathbf{1} \otimes \sigma^3 \otimes \sigma^2 \\
\gamma^4 &= \sigma^2 \otimes \sigma^1 \otimes \sigma^2 \otimes \mathbf{1} \\
\gamma^5 &= \sigma^2 \otimes \sigma^3 \otimes \sigma^2 \otimes \mathbf{1} \\
\gamma^6 &= \sigma^2 \otimes \sigma^2 \otimes \mathbf{1} \otimes \sigma^1 \\
\gamma^7 &= \sigma^2 \otimes \sigma^2 \otimes \mathbf{1} \otimes \sigma^3 \\
\gamma^8 &= \sigma^2 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}
\end{aligned}$$

and the 8-dimensional chirality operator is given by

$$\gamma^{11} = \sigma^3 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}.$$

These γ^m are all real and symmetric. The charge conjugation matrix is $C = \Gamma^0$. Notice that $\Gamma^{11} \equiv \prod_{m=0}^{9} \Gamma^m = \sigma^3 \otimes \mathbf{1}_{16}$, which is the 10-dimensional chirality operator. The Γ-matrices satisfy the following algebra

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}.$$

A.3 The Killing Spinor Equations

We arrive at the Killing spinor equations by requiring that the supersymmetry variations of the fermions vanish.

A.3.1 Type IIA String Theory

The Killing spinor variations for type IIA string theory are written in terms of the 32-component full spinor E. The gravitino variation is

$$\delta\Phi_m = \nabla_m E - \frac{1}{8}\not{H}_m\Gamma_{11}E - \frac{1}{16}e^{\phi}\not{F}_{(2)}\Gamma_m\Gamma_{11}E + \frac{1}{192}e^{\phi}\not{F}_{(4)}\Gamma_m E,$$

and the dilatino variation is

$$\delta\Lambda = \left(\not{\partial}\phi - \frac{1}{12}\not{H}\Gamma_{11} - \frac{3}{8}e^{\phi}\not{F}_{(2)}\Gamma_{11} + \frac{1}{96}e^{\phi}\not{F}_{(4)}\right)E.$$

A.3.2 Type IIB String Theory

Type IIB string theory contains fermions which are doublets of gravitini and dilatini, which have opposite chirality. In this treatment, we choose the dilatini λ and $\hat{\lambda}$ to have negative chirality, whilst the gravitini ψ and $\hat{\psi}$ have positive chirality. The Killing spinors, or supersymmetry parameters, ϵ and $\hat{\epsilon}$ also have positive chirality. Then the gravitino variations (in two-component form) are

$$\begin{aligned}\delta\psi_m =& \nabla_m\epsilon - \frac{1}{4}\not{H}_m\epsilon - \frac{e^{\phi}}{8}\left(\not{F}_{(1)} + \not{F}_{(3)} + \frac{1}{2}\not{F}_{(5)}\right)\Gamma_m\hat{\epsilon},\\ \delta\hat{\psi}_m =& \nabla_m\hat{\epsilon} + \frac{1}{4}\not{H}_m\hat{\epsilon} + \frac{e^{\phi}}{8}\left(\not{F}_{(1)} - \not{F}_{(3)} + \frac{1}{2}\not{F}_{(5)}\right)\Gamma_m\epsilon,\end{aligned}$$

and the dilatino variations (in two-component form) are

$$\begin{aligned}\delta\lambda =& \not{\partial}\phi\epsilon - \frac{1}{2}\not{H}\epsilon + \frac{e^{\phi}}{2}\left(2\not{F}_{(1)} + \not{F}_{(3)}\right)\hat{\epsilon},\\ \delta\hat{\lambda} =& \not{\partial}\phi\hat{\epsilon} + \frac{1}{2}\not{H}\hat{\epsilon} - \frac{e^{\phi}}{2}\left(2\not{F}_{(1)} - \not{F}_{(3)}\right)\epsilon,\end{aligned}$$

where

$$\begin{aligned}\not{F}_{(n)} =& \frac{1}{n!}F_{m_1\ldots m_n}\Gamma^{m_1\ldots m_n}\\ \not{H}_m =& \frac{1}{2}H_{mnr}\Gamma^{nr}.\end{aligned}$$

A.4 Hodge Duality

The Hodge duality conventions are such that on a *p-form* in *D*-dimensions

$$(\star F_p)_{\mu_{p+1}\cdot\mu_D} = \frac{1}{p!}\sqrt{|g|}\epsilon_{\mu_1\ldots\mu_D}F^{\mu_1\ldots\mu_p}.$$

Note also that $\star\star F_p = s(-1)^{p(D-1)} F_p$ where s is the spacetime signature.

Appendix B

$D(2,1;\alpha)$ Supercoset Currents

This appendix is taken directly from our paper [36].

Due to the structure (5.14) of the superalgebra $\mathfrak{d}(2,1;c^2)$ and the chosen coset element (5.15), the coset currents $J^{(0)}$ and J have the following components (see (5.19)):

$$\begin{aligned}
J_P^{(0)} &= \left[\mathrm{e}^{-B}\left(\mathrm{d}xP + \mathrm{d}\theta^\alpha Q_\alpha + \mathrm{d}\lambda_+ L_+\right)\mathrm{e}^B\right]_P , \quad J_K^{(0)} = 0 , \quad J_D^{(0)} = \left[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^B\right]_D , \\
J_{L_+}^{(0)} &= \left[\mathrm{e}^{-B}\left(\mathrm{d}xP + \mathrm{d}\theta^\alpha Q_\alpha + \mathrm{d}\lambda_+ L_+\right)\mathrm{e}^B\right]_{L_+} , \quad J_{L_-}^{(0)} = 0 , \quad J_{L_3}^{(0)} = \left[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^B\right]_{L_3} , \\
J_{R^\alpha{}_\beta}^{(0)} &= \left[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^B\right]_{R^\alpha{}_\beta} , \\
J_{Q_\alpha}^{(0)} &= \left[\mathrm{e}^{-B}\left(\mathrm{d}xP + \mathrm{d}\theta^\alpha Q_\alpha + \mathrm{d}\lambda_+ L_+\right)\mathrm{e}^B\right]_{Q_\alpha} , \quad J_{\hat{Q}_\alpha}^{(0)} = \left[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^B\right]_{\hat{Q}_\alpha} , \\
J_{S_\alpha}^{(0)} &= 0 , \quad J_{\hat{S}_\alpha}^{(0)} = \left[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^B\right]_{\hat{S}_\alpha}
\end{aligned} \tag{B.1a}$$

and

$$\begin{aligned}
J_P &= J_P^{(0)} , \quad J_{Q_\alpha} = J_{Q_\alpha}^{(0)} , \quad J_{L_+} = J_{L_+}^{(0)} , \\
J_D &= J_D^{(0)} + \sigma^2_{\alpha\beta} J_{Q_\alpha}^{(0)}\xi^\beta , \quad J_{L_3} = J_{L_3}^{(0)} + \mathrm{i}c^2\sigma^2_{\alpha\beta}J_{Q_\alpha}^{(0)}\xi^\beta , \\
J_{R^\alpha{}_\beta} &= J_{R^\alpha{}_\beta}^{(0)} - \mathrm{i}s^2\sigma^2_{\alpha\gamma}J_{Q_\gamma}^{(0)}\xi^\beta - \tfrac{\mathrm{i}}{2}s^2\sigma^2_{\gamma\delta}\delta^\alpha{}_\beta J_{Q_\gamma}^{(0)}\xi^\delta , \\
J_{\hat{Q}_\alpha} &= J_{\hat{Q}_\alpha}^{(0)} - J_P^{(0)}\xi^\alpha , \quad J_{\hat{S}_\alpha} = J_{\hat{S}_\alpha}^{(0)} + J_{L_+}^{(0)}\xi^\alpha , \\
J_K &= -\sigma^2_{\alpha\beta}J_{\hat{S}_\alpha}^{(0)}\xi^\beta + \tfrac{\mathrm{i}}{2}J_{L_+}^{(0)}\xi^2 , \quad J_{L_-} = -c^2\sigma^2_{\alpha\beta}J_{\hat{Q}_\alpha}^{(0)}\xi^\beta - \tfrac{\mathrm{i}}{2}c^2 J_P^{(0)}\xi^2 , \\
J_{S_\alpha} &= -\tfrac{1}{2}J_D^{(0)}\xi^\alpha - \tfrac{\mathrm{i}}{2}J_{L_3}^{(0)}\xi^\alpha - \mathrm{i}J_{R^\beta{}_\alpha}^{(0)}\xi^\beta + \mathrm{d}\xi^\alpha - \tfrac{\mathrm{i}}{2}s^2 J_{Q_\alpha}^{(0)}\xi^2 .
\end{aligned} \tag{B.1b}$$

In these expressions, we have made all the ξ-dependence explicit.

Dual currents

The Maurer-Cartan form $\tilde{J} = \tilde{g}^{-1}\mathrm{d}\tilde{g}$ constructed from the dual coset representative (5.26) is of the form

$$\begin{aligned}
\tilde{J} &= \mathrm{e}^{-\sigma^{1\beta}{}_\alpha\xi^\alpha Q_\beta}\tilde{J}^{(0)}\mathrm{e}^{\sigma^{1\beta}{}_\alpha\xi^\alpha Q_\beta} + \sigma^{1\,\beta}{}_\alpha\mathrm{d}\xi^\alpha Q_\beta \\
&= \tilde{J}^{(0)} - \sigma^{1\,\beta}{}_\alpha\xi^\alpha\left[S_\beta, \tilde{J}^{(0)}\right] - \tfrac{\mathrm{i}}{4}\xi^2\sigma^{2\,\alpha\beta}\left\{S_\alpha, \left[S_\beta, \tilde{J}^{(0)}\right]\right\} + \sigma^{1\,\beta}{}_\alpha\mathrm{d}\xi^\alpha Q_\beta ,
\end{aligned} \tag{B.2}$$

where, as before, $\tilde{J}^{(0)}$ does not depend on the fermonic coordinate ξ^α, and we have set $\xi^2 := \mathrm{i}\sigma^2_{\alpha\beta}\xi^\alpha\xi^\beta$. A calculation similar to the one that led to (B.1) yields

$$\begin{aligned}
\tilde{J}^{(0)}_P &= 0\ ,\quad \tilde{J}^{(0)}_K = \big[\mathrm{e}^{-B}\big(\mathrm{d}\tilde{x}K - \mathrm{i}\sigma^{2\,\alpha\beta}\mathrm{d}\tilde{\theta}_\alpha S_\beta + \mathrm{d}\tilde{\lambda}_+ L_-\big)\mathrm{e}^B\big]_K\ ,\quad \tilde{J}^{(0)}_D = \big[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^B\big]_D\ ,\\
\tilde{J}^{(0)}_{L_+} &= 0\ ,\quad \tilde{J}^{(0)}_{L_-} = \big[\mathrm{e}^{-B}\big(\mathrm{d}\tilde{x}K - \mathrm{i}\sigma^{2\,\alpha\beta}\mathrm{d}\tilde{\theta}_\alpha S_\beta + \mathrm{d}\tilde{\lambda}_+ L_-\big)\mathrm{e}^B\big]_{L_-}\ ,\quad \tilde{J}^{(0)}_{L_3} = \big[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^B\big]_{L_3}\ ,\\
\tilde{J}^{(0)}_{R^\alpha{}_\beta} &= \big[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^B\big]_{R^\alpha{}_\beta}\ ,\\
\tilde{J}^{(0)}_{Q_\alpha} &= 0\ ,\quad \tilde{J}^{(0)}_{\hat{Q}_\alpha} = \big[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^B\big]_{\hat{Q}_\alpha}\ ,\\
\tilde{J}^{(0)}_{S_\alpha} &= \big[\mathrm{e}^{-B}\big(\mathrm{d}\tilde{x}K - \mathrm{i}\sigma^{2\,\alpha\beta}\mathrm{d}\tilde{\theta}_\alpha S_\beta + \mathrm{d}\tilde{\lambda}_+ L_-\big)\mathrm{e}^B\big]_{S_\alpha}\ ,\quad \tilde{J}^{(0)}_{\hat{S}_\alpha} = \big[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^B\big]_{\hat{S}_\alpha}
\end{aligned} \tag{B.3a}$$

and

$$\begin{aligned}
\tilde{J}_K &= \tilde{J}^{(0)}_K\ ,\quad \tilde{J}_{S_\alpha} = \tilde{J}^{(0)}_{S_\alpha}\ ,\quad \tilde{J}_{L_-} = \tilde{J}^{(0)}_{L_-}\ ,\\
\tilde{J}_D &= \tilde{J}^{(0)}_D - \mathrm{i}\sigma^1_{\alpha\beta}\tilde{J}^{(0)}_{S_\alpha}\xi^\beta\ ,\quad \tilde{J}_{L_3} = \tilde{J}^{(0)}_{L_3} + c^2\sigma^1_{\alpha\beta}\tilde{J}^{(0)}_{S_\alpha}\xi^\beta\ ,\\
\tilde{J}_{R^\alpha{}_\beta} &= \tilde{J}^{(0)}_{R^\alpha{}_\beta} + s^2\big(\sigma^1_{\alpha\gamma}\tilde{J}^{(0)}_{S_\beta} - \tfrac{1}{2}\delta^\beta{}_\alpha\sigma^1_{\gamma\delta}\tilde{J}^{(0)}_{S_\delta}\big)\xi^\gamma\ ,\\
\tilde{J}_{\hat{Q}_\alpha} &= \tilde{J}^{(0)}_{\hat{Q}_\alpha} + \sigma^{1\,\alpha}{}_\beta\tilde{J}^{(0)}_{L_-}\xi^\beta\ ,\quad \tilde{J}_{\hat{S}_\alpha} = \tilde{J}^{(0)}_{\hat{S}_\alpha} - \sigma^{1\,\alpha}{}_\beta\tilde{J}^{(0)}_K\xi^\beta\ ,\\
\tilde{J}_P &= -\mathrm{i}\sigma^1_{\alpha\beta}\tilde{J}^{(0)}_{\hat{Q}_\alpha}\xi^\beta - \tfrac{\mathrm{i}}{2}\tilde{J}^{(0)}_{L_-}\xi^2\ ,\quad \tilde{J}_{L_+} = -\mathrm{i}c^2\sigma^1_{\alpha\beta}\tilde{J}^{(0)}_{\hat{S}_\alpha}\xi^\beta + \tfrac{\mathrm{i}}{2}c^2\tilde{J}^{(0)}_K\xi^2\ ,\\
\tilde{J}_{Q_\alpha} &= \sigma^{1\,\alpha}{}_\beta\big(\tfrac{1}{2}\tilde{J}^{(0)}_D\xi^\beta + \tfrac{\mathrm{i}}{2}\tilde{J}^{(0)}_{L_3}\xi^\beta + \mathrm{d}\xi^\beta\big) - \mathrm{i}\sigma^{1\,\beta}{}_\gamma\tilde{J}^{(0)}_{R^\beta{}_\alpha}\xi^\gamma + \tfrac{\mathrm{i}}{2}s^2\tilde{J}^{(0)}_{S_\alpha}\xi^2\ .
\end{aligned} \tag{B.3b}$$

As before, in these expressions, we have made all the ξ-dependence explicit. Note that $\tilde{J}^{(0)}_{\hat{Q}_\alpha} = J^{(0)}_{\hat{Q}_\alpha}$, $\tilde{J}^{(0)}_{\hat{S}_\alpha} = J^{(0)}_{\hat{S}_\alpha}$, $\tilde{J}^{(0)}_D = J^{(0)}_D$, $\tilde{J}^{(0)}_{L_3} = J^{(0)}_{L_3}$, and $\tilde{J}^{(0)}_{R^\beta{}_\alpha} = J^{(0)}_{R^\beta{}_\alpha}$, respectively.

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